Vector bundles on degenerations of elliptic curves of types II, III and IV

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Abstract

In this thesis we classify simple coherent sheaves on Kodaira fibers of types II, III and IV (cuspidal and tacnode cubic curves and a plane configuration of three concurrent lines).

Indecomposable vector bundles on smooth elliptic curves were classified in 1957 by Atiyah. In works of Burban, Drozd and Greuel it was shown that the categories of vector bundles and coherent sheaves on cycles of projective lines are tame.

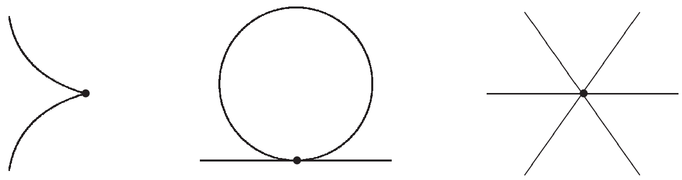
It turns out, that all other degenerations of elliptic curves are vector-bundle-wild. Nevertheless, we prove that the category of coherent sheaves of an arbitrary reduced plane cubic curve, (including the mentioned Kodaira fibers) is brick-tame. The main technical tool of our approach is the representation theory of bocses. Although, this technique was mainly used for purely theoretical purposes, we illustrate its computational potential for investigating tame behavior in wild categories. In particular, it allows to prove that a simple vector bundle on a reduced cubic curve is determined by its rank, multidegree and determinant, generalizing Atiyah's classification. Our approach leads to an interesting class of bocses, which can be wild but are brick-tame.

Contents

Chapter 1

Introduction

This PhD thesis deals with the problem of classification of simple [1] vector bundles and torsion free sheaves on plane cubic projective curves given by the equations $zy^2 = x^3$ (cuspidal curve), $x(yz - x^2) = 0$ (tacnode curve) and $xy(x + y) = 0$ (three concurrent lines).



In terms of Kodaira's list of degenerations of elliptic curves these are the Kodaira fibers of types II, III and IV respectively, see for example [BPV84].

Our main source of inspiration is the following classical result of Atiyah:

Theorem 1.0.1 ([Ati57]). *Let E be a smooth elliptic curve over an algebraically closed field $\Bbbk$. Then:*

- *a simple vector bundle $\mathcal{E}$ on E is uniquely determined by its rank r, degree d, which should be coprime, and determinant $det(\mathcal{E}) \in \mathrm{Pic}^d(E) \cong E$;*

- *an indecomposable vector bundle $\mathcal{F}$ of rank r and degree d can be described in a unique way by $\mathcal{F} \cong \mathcal{E} \otimes \mathcal{A}_h$, where $h := g.c.d.(r, d)$, $\mathcal{E}$ is a simple vector bundle of rank r/h and degree d/h and $\mathcal{A}_h$ is an indecomposable vector bundle, recursively defined by the sequences*

$$0 \longrightarrow \mathcal{A}_{h-1} \longrightarrow \mathcal{A}_h \longrightarrow \mathcal{O} \longrightarrow 0, \quad h \geq 2, \quad \mathcal{A}_1 = \mathcal{O}.$$

Our work grew up from an attempt to generalize this theorem to the case of degenerations of elliptic curves listed above. It is related to and is partially motivated by the following problems.

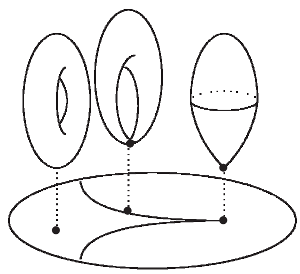
[1] A sheaf is called simple if it admits no endomorphisms but homotheties.

Motivation

- Tame behavior in wild categories. It turns out that the problem of classification of *all indecomposable* vector bundles on Kodaira fibers of types II, III and IV is representation-wild, meaning that it contains as a subproblem a classification of all finite-dimensional representations of an *arbitrary* finitely generated $\Bbbk$–algebra. However, in this work we show that a description of *simple* vector bundles and torsion free sheaves can be reduced to a problem of linear algebra of tame representation type.

- Applications in mathematical physics. In the work of Polishchuk [Pol02] it was shown that the study of simple vector bundles on a Gorenstein projective curve E with trivial canonical bundle is closely related to the theory of the classical and quantum Yang-Baxter equations. To study degenerations of solutions of classical Yang-Baxter equation it is necessary to compute explicitly certain triple Massey products in the derived category of coherent sheaves $\mathcal{D}^b(\mathsf{Coh}_E)$. It turns out that the standard technique of moduli spaces and of Fourier-Mukai transforms is not sufficient to carry out these computations and the approach via matrix problems provides the right tool to deal with this problem, see [BK4].

- Vector bundles on elliptic fibrations. Our description of simple vector bundles on degenerate elliptic curves contributes to a better understanding of the theory of vector bundles on elliptic fibrations.

  Indeed, elliptically fibered varieties arising in algebraic geometry usually have singular fibers and one can study invariants of vector bundles on them by looking at their restrictions to singular fibers.

- Representations of bocses. The key idea of our approach is to reduce the classification of vector bundles on a degenerate elliptic curve to a certain problem of linear algebra (a matrix problem). In this work we deal with curves for which all indecomposable vector bundles can not be classified in the sense of representation theory (vector-bundle-wild curves); however, it is possible to describe simple vector bundles. This problem can be naturally interpreted in the language of bocses (bimodules over categories with a coalgebra structure). A famous result in the representation theory of algebras and bocses is Drozd's Tame-Wild dichotomy theorem [Dro79], see also [CB88]. There is a conjecture saying that an analogous statement should hold for bricks (i.e. objects with no nonscalar endomorphisms). This is still not proven in general; however, our approach to study vector bundles on degenerations of singular curves leads to a wide class of bocses which are representation wild but brick-tame.

1.1 History of the subject

In 1908 Birkhoff [Bir13] proved that any invertible matrix $M \in \mathrm{GL}(n, \Bbbk[t, t^{-1}])$ can be diagonalized using the transformation rule $M \mapsto S^{-1}MT$, where $S \in \mathrm{GL}(n, \Bbbk[t])$ and $T \in \mathrm{GL}(n, \Bbbk[t^{-1}])$: $M \sim \mathrm{diag}(t^{m_1}, \dots, t^{m_n})$. Moreover, the integers $m_1, \dots, m_n$ are uniquely determined up to a permutation. In modern language this can be rephrased as

Theorem 1.1.1. *A vector bundle $\mathcal{E}$ on the projective line $\mathbb{P}^1$ splits into a direct sum of line bundles:*

$$\mathcal{E} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(m_i).$$

This theorem was rediscovered in 1955 by Grothendieck [Gro57]. Two years later Atiyah proved Theorem 1.0.1 for an elliptic curve E. In 1971 Oda gave a more explicit description of indecomposable vector bundles on one-dimensional complex tori in terms of étale coverings [Oda71]. Further results about a moduli spaces of semistable vector bundles on E were obtained in [Tu93]. Long attempts to generalize Atiyah's result either to curves of higher genus or to abelian varieties of higher dimension moved the barycentre of the research towards the study of *stable bundles* and their *moduli spaces*.

Moduli spaces of vector bundles on curves

Starting from the 1970s the main attention has been paid to a study of moduli spaces of vector bundles on smooth projective varieties (see for example [LeP97, New78, Ses82]). There are also many results about the existence of moduli spaces of vector bundles on irreducible reduced curves (see [New78, Ses82, Bho92]). However, besides some trivial cases it is still not much known about vector bundles on curves with many components.

The study of vector bundles began with the study of the Picard group of an algebraic variety. In [Gro62] Grothendieck showed that for any irreducible reduced curve X there exists a fine moduli space $\mathscr{J}$ of the functor Pic^d of line bundles of degree d, and the space of parameters, usually called the *generalized Jacobian*, has dimension $p_a(X)$, which is the arithmetic genus of X.

However, if X is singular, then the moduli space of line bundles of degree d in general is not projective but only quasi-projective. Compactifications of the Picard scheme have been studied by many authors, we mention only some of them. In [D'S79] D'Souza showed that there exists a natural compactification of $\mathscr{J}$, and this compactification is the moduli space of torsion free sheaves $\bar{\mathscr{J}}$. In [AK80, AK79] Altman and Kleiman investigated further properties of the compactified Jacobian, in particular, its behavior in a relative situation. Especially, from Theorem 8.8 of [AK80], it follows that for an irreducible and reduced curve E of arithmetic genus one, the compactified Jacobian $\overline{\mathsf{Pic}}^0$ is represented by the scheme E itself.

Analogously as for a smooth curve X there is a course moduli space $\mathscr{M}$ of the moduli functor $\mathsf{VB}^s_X(r,d)$ which is fine provided that $g.c.d(r,d) = 1$, it was shown in [New78, Theorem 5.8′] for an irreducible and reduced curve X that there is a coarse moduli space $\bar{\mathscr{M}}$ of the moduli functor $\mathsf{TF}^s_X(r,d)$ of stable torsion free sheaves of rank r and degree d and there is a natural compactification of $\bar{\mathscr{M}}$ by adding classes of semi-stable torsion free sheaves[2]. Moreover, if r and d are coprime, then $\bar{\mathscr{M}}$ is a fine moduli space (see Theorem 5.12′).

For a curve with a singularity different from an ordinary node it is still not much known about vector bundles. For curves of arithmetic genus one we should mention a resent result of López-Martin. In [Lo05] she described the geometry of the compactified Simpson Jacobian for Kodaira fibers I_N, II, III and IV and in [Lo06] considered the relative situation.

Generalized parabolic bundles (GPB)

For a rather long time (till the middle of the 70s) there were no efficient methods to study moduli spaces of vector bundles of higher ranks on singular curves. It was Seshadri who proposed for this purpose the method of parabolic bundles (PB) (see Section 3 of [Ses82]). Bhosle generalized the method considering parabolic structures over divisors. The main idea of this approach is as follows. Let X be a reduced singular curve, with nodes $p_1, \ldots, p_n$ and cusps $q_1, \ldots, q_m$ as singularities. Let $\widetilde{X} \xrightarrow{\pi} X$ be its normalization, $\pi^{-1}(p_i) = \{\tilde{p}_i, \tilde{p}'_i\}$ and $\pi^{-1}(q_j) = \{\tilde{q}_j\}$. Let $\widetilde{S}$ denote the scheme-theoretic preimage of the singular locus S defined by the conductor i.e. as a divisor

$$\widetilde{S} = \sum_{i=1}^{n} (\tilde{p}_i + \tilde{p}'_i) + \sum_{j=1}^{m} 2\tilde{q}_j.$$

[2] In what follows by compactification we always mean the compactification of moduli spaces of stable vector bundles by stable torsion free sheaves.

Consider a vector bundle $\widetilde{\mathcal{E}}$ of rank r on the normalization $\widetilde{X}$ together with a parabolic structure on the divisor $\widetilde{S}$ which is a set of vector subspaces of rank r

$$F_{p_i}(\widetilde{\mathcal{E}}) \xhookrightarrow{(\eta,\eta')} \left(\widetilde{\mathcal{E}} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{S}}|_{\tilde{p}_i}\right) \oplus \left(\widetilde{\mathcal{E}} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{S}}|_{\tilde{p}'_i}\right)$$

for $1 \leq i \leq n$, such that both maps η and η' are isomorphisms; and

$$F_{q_j}(\widetilde{\mathcal{E}}) \xhookrightarrow{\zeta} \widetilde{\mathcal{E}} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{S}}|_{2\tilde{q}_j}$$

for $1 \leq j \leq m$, such that the induced map $F_{q_j}(\widetilde{\mathcal{E}}) \to \widetilde{\mathcal{E}} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{S}}|_{\tilde{q}_j}$ is an isomorphism. Hence, there is a vector subspace

$$F(\widetilde{\mathcal{E}}) \hookrightarrow \widetilde{\mathcal{E}} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{S}} =: \widetilde{\mathcal{E}}(\widetilde{S}).$$

The pair $(\widetilde{\mathcal{E}}, F(\widetilde{\mathcal{E}}))$ is called a *generalized parabolic bundle* (GPB). If the curve X is irreducible then for a GPB one can easily introduce the notions of degree and stability. The category of stable GPB's of rank r and degree d is denoted by $\mathsf{GPB}^s_X(r,d)$. A coarse moduli space $\widetilde{\mathscr{M}}$ of stable of GPB's can be constructed using methods of geometric invariant theory (for instance see [Bho92, Theorem 1]). If r and d are coprime then $\widetilde{\mathscr{M}}$ is fine. For a generalized parabolic bundle $(\widetilde{\mathcal{E}}, F(\widetilde{\mathcal{E}}))$ on $\widetilde{X}$ consider the canonical map

$$\varphi : \widetilde{\mathcal{E}} \twoheadrightarrow \widetilde{\mathcal{E}}(\widetilde{S}) \twoheadrightarrow \widetilde{\mathcal{E}}(\widetilde{S})/F(\widetilde{\mathcal{E}}).$$

Let TF_X be the category of torsion free sheaves on X and $\mathsf{TF}^s_X(r,d)$ its full subcategory of stable torsion free sheaves of rank r and degree d. Define a functor $\Phi : \mathsf{GPB}_X \to \mathsf{TF}_X$, taking $(\widetilde{\mathcal{E}}, F(\widetilde{\mathcal{E}})) \mapsto \ker(\pi_*\varphi)$. [Bho96, Theorem 4] asserts that Φ maps $\mathsf{GPB}^s_X(r,d)$ to $\mathsf{TF}^s_X(r,d)$ and and the induced map $\phi : \widetilde{\mathscr{M}} \to \mathscr{M}$ has the following properties: ϕ is surjective, it is an isomorphism on $\phi^{-1}(\mathscr{M})$; and if $g.c.d.(r,d) = 1$ then $\widetilde{\mathscr{M}}$ is the normalization of $\mathscr{M}$. In particular, $(\widetilde{\mathcal{E}}, F(\widetilde{\mathcal{E}}))$ is stable if and only if its image $\mathcal{F}$ is stable. The method can be also used for curves with many components (see [Bho93]).

Fourier-Mukai transforms

A special interest in the theory of vector bundles on singular curves of arithmetic genus one cames back in the 1990s by work of Friedman, Morgan and Witten on vector bundles on elliptic fibrations [FMW99] (see also [Teo00]). They discovered a method to construct relatively semi-stable torsion free sheaves of degree zero in terms of so-called spectral coverings. Because of the importance of this construction in string theory it was studied by many authors. Recently, it was put in a general framework of Fourier-Mukai transforms see [BBHM02, BK05] and [BK06]. The idea behind the Fourier-Mukai approach can be explained as follows. Let E be a reduced and irreducible curve of arithmetic genus one and $p_0 \in E_{reg}$ be a fixed smooth point on it. Let $\mathsf{TF}_E^{ss}(0)$ denote the category of semi-stable torsion free sheaves of degree zero on E, and let Tor_E be the category of torsion sheaves (i.e. skyscraper sheaves) on E. For a sheaf $\mathcal{F} \in \mathsf{TF}_E^{ss}(0)$, consider the natural evaluation map

$$ev : H^0(\mathcal{F} \otimes \mathcal{O}(p_0)) \otimes_{\Bbbk} \mathcal{O} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(p_0).$$

The map ev turns out to be an injective morphism of two torsion free sheaves of the same rank, thus $\operatorname{coker}(ev) \in \mathsf{Tor}_E$. Then the functor

$$\begin{aligned} \mathbb{F} : \mathsf{TF}_E^{ss}(0) &\to \mathsf{Tor}_E \\ \mathcal{F} &\mapsto \operatorname{coker}(ev). \end{aligned} \tag{1.1}$$

is an exact equivalence of abelian categories (see [FMW99] Theorem 1.2 and [Teo00] Theorem 1.3). Hence, semi-stable torsion free sheaves of degree zero can be indirectly described by skyscraper sheaves. For a semi-stable torsion free sheaf of rank r and degree d the functor

$$\begin{aligned} \mathsf{T}_{\Bbbk(p_0)} := - \otimes \mathcal{O}(p_0) : \mathsf{TF}_E^{ss} &\to \mathsf{TF}_E^{ss} \\ \mathcal{F} &\mapsto \mathcal{F} \otimes \mathcal{O}(p_0). \end{aligned} \tag{1.2}$$

preserves r and raises the degree to $d+r$. If we had an equivalence $\mathbb{F} : \mathsf{TF}_E^{ss} \to \mathsf{TF}_E^{ss}$ mapping $\mathsf{TF}_E^{ss}(r,d)$ to $\mathsf{TF}_E^{ss}(r+d,d)$, then concatenating $\mathbb{F}$ and $\mathsf{T}_{\Bbbk(p_0)}$ we could recover a description of semi-stable torsion free sheaves of arbitrary coprime rank and degree from a description of torsion sheaves. A functor $\mathbb{F}$ with such properties can be naturally constructed on the level of derived categories. This approach is highly efficient but it requires some advanced machinery of homological algebra. For an irreducible curve of arithmetic genus one, the autoequivalence $\mathbb{F}$ transforms semi-stable sheaves to semi-stable ones. For the smooth case see e.g., Theorem 14.7 [Pol03]; for a general case see Theorem 4.1 [BK3]. Unfortunately, this method works well only for irreducible curves of arithmetic genus one. From the beginning of the last decade the idea to apply the technique of derived categories to study geometric problems became rather popular. Nowadays it is developing rapidly. However, there is still not much known about the derived category of coherent sheaves on Gorenstein projective curves with many components. So far this method has been applied only to irreducible and reduced curves of arithmetic genus one.

Matrix problem approach

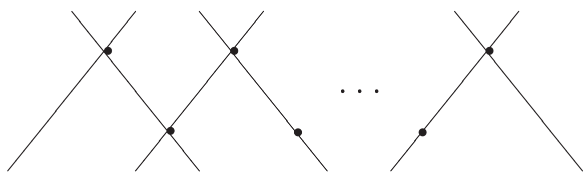
Another method for dealing with vector bundles on an arbitrary reduced singular projective curve was suggested by Drozd and Greuel [DG01]. The main idea of their approach can be explained as follows. Let X be a singular reduced projective curve (typically rational, but with arbitrary singularities), $\pi : \tilde{X} \to X$ its normalization. Then a description of the fibers of the functor $\pi^* : \mathsf{VB}_X \to \mathsf{VB}_{\tilde{X}}$ can be converted to some representation theory problem, called a matrix problem.

In order to make our further discussions more comprehensible we shortly explain the main ideas of the matrix problem technique, which is a powerful tool for studying a wide class of classification problems. Consider a Krull-Schmidt category, whose objects M are matrices or tuples of matrices, and the set of morphisms $F : M \to M'$ is given by a collection of rules (called transformations) saying how M can be modified. A problem of finding a *canonical form* of an arbitrary indecomposable object M with respect to admissible transformations F is called a *matrix problem* or a *linear algebra problem.* Rigorously speaking, it is a problem of describing indecomposable (respectively simple, rigid) objects of the category of representations of some differential biquiver (Q, ∂). Actually, it always boils down to the classification of orbits of indecomposable objects under the action as explained above. The simplest examples are the problems of finding a Gauß form of a matrix (transformations are $M \mapsto SMT$, for $M \in \mathrm{Mat}_{\Bbbk}(r_1 \times r_2)$, $S \in \mathrm{GL}(r_1, \Bbbk)$ and $T \in \mathrm{GL}(r_2, \Bbbk)$) and a Jordan normal form of a square matrix (transformations $M \mapsto S^{-1}MS$, for $M \in \mathrm{Mat}_{\Bbbk}(r \times r)$ and $S \in \mathrm{GL}(r, \Bbbk)$). One reduces a matrix to its canonical form by a very general algorithm called *matrix reduction.* Obviously, there are cases when a canonical form does not exist. Such "nasty" problems are called *wild*, a classical example is the question about a normal form of a pair of non-commuting matrices using simultaneous conjugations

$$S : (M_1, M_2) \mapsto (S^{-1}M_1S, S^{-1}M_2S),$$

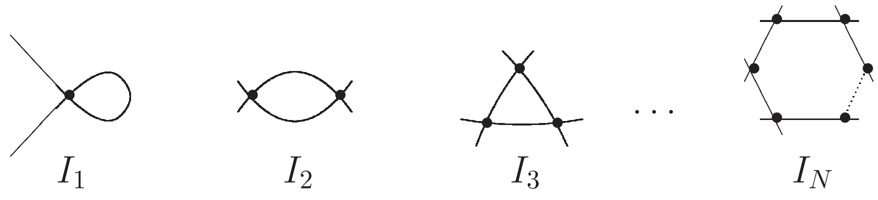
where $S \in \mathrm{GL}(\Bbbk, r)$ and $M_1, M_2 \in \mathrm{Mat}_{\Bbbk}(r \times r)$, or equivalently, a description of indecomposable finite dimensional representations of the free algebra $\Bbbk\langle x, y\rangle$. However, in some cases a matrix problem can be completely solved then it is called *tame.*

Coming back to our original question about vector bundles on singular curves, if the corresponding matrix problem is tame and we have a canonical form then one can interpret the result in terms of sheaves. The approach of Drozd and Greuel is quite similar to the method of generalized parabolic bundles. For example, both of them allow us to generalize Birkhoff-Grothendick's theorem to the case of a chain of projective lines (see Appendix A)



Theorem 1.1.2 ([Bho93, DG01]). *Let X be a chain of projective lines, then any vector bundle on X splits into a direct sum of line bundles.*

The main applications of the Drozd-Greuel method concern the case of curves of arithmetic genus one. In the case of a cycle of N projective lines (Kodaira cycles I_N)



one obtains the following classification of indecomposable torsion free sheaves. Let I_N be a cycle of N projective lines and C_k a chain of k projective lines.

Theorem 1.1.3 ([DG01], see also [BBDG]). *Let $\mathcal{E}$ be an indecomposable torsion free sheaf on I_N.*

1. *If $\mathcal{E}$ is locally free, then there is an étale covering $\pi_k : I_{Nk} \longrightarrow I_N$, a line bundle $\mathcal{L} \in \mathrm{Pic}(I_{Nk})$ and a natural number $h \in \mathbb{N}$ such that*
$$\mathcal{E} \cong \pi_{k*}(\mathcal{L}) \otimes \mathcal{A}_h,$$
where $\mathcal{A}_h$ is an indecomposable vector bundle, recursively defined by the sequences
$$0 \longrightarrow \mathcal{A}_{h-1} \longrightarrow \mathcal{A}_h \longrightarrow \mathcal{O} \longrightarrow 0, \quad h \geq 2, \quad \mathcal{A}_1 = \mathcal{O}.$$
2. *If $\mathcal{E}$ is not locally free, then there exists $k \in \mathbb{N}$, a map $p_k : C_k \longrightarrow I_N$ and a line bundle $\mathcal{L} \in \mathrm{Pic}(C_k)$ such that $\mathcal{E} \cong p_{k*}(\mathcal{L})$.*

This classification is completely parallel to Oda's description of vector bundles on elliptic curves [Oda71] and provides quite simple rules for the computation of a decomposition of the tensor product of any two vector bundles into a direct sum of indecomposable ones. It allows one to compute the dual bundle of an indecomposable vector bundle as well as dimensions of homomorphism and extension spaces between indecomposable vector bundles (and in particular, their cohomology), see [Bur03, BBDG].

The proof of Theorem 1.1.3 essentially uses ideas coming from representation theory and the technique of matrix problems [Bon92] (see also [KL86], [CB89]). Using a similar approach this result was generalized by Burban and Drozd [BD04] to a classification of indecomposable complexes as objects of the derived category bounded from the right of coherent sheaves $\mathcal{D}^-(\mathsf{Coh}_E)$ on a cycle of projective lines $E = I_N$.

However, a description of simple vector bundles on E requires some extra work. In the case of a rational curve with one node a simple vector bundle is determined by its rank and degree (which have to be coprime) and one continuous parameter $\lambda \in \mathbb{k}^*$, see [Bur03]. Hence, in this case the combinatorics of the answer is completely parallel to the case of an elliptic curve.

Vector-bundle-wild curves

Although there is a complete classification of indecomposable torsion free sheaves on Kodaira cycles, the situation turns out to be quite different for the other singular projective curves of arithmetic genus one. For example, for a cuspidal rational curve $zy^2 = x^3$ even the classification of indecomposable semi-stable vector bundles of a given slope is a *representation-wild* problem. However, if we additionally impose the simplicity assumption, then the wild fragments of the corresponding matrix problem disappear and we can reduce the matrices to a canonical form. Especially, for a cuspidal cubic curve such a problem is analogous to a problem already considered by Drozd in his study of representations of mixed Lie Groups [Dro92].

The matrix problem for simple vector bundles on a cuspidal cubic curve is relatively easy to deal with, since it is *self-reproducing*, i.e. after applying one step of matrix reduction we obtain the same problem but with matrices of smaller sizes. In fact, the matrix reduction operates on discrete parameters of a vector bundle as Euclidean algorithm. Carrying this out, we obtain a description of simple vector bundles on a cuspidal cubic curve, which is quite parallel to the first part of Atiyah's Theorem 1.0.1. Analogous results can be obtained by applying the GPB approach or Fourier-Mukai technique. The main advantage of our description is that it is explicit and can be used for further applications mentioned earlier, in particular for a calculation of Massey products. For other curves of arithmetic genus one treated in this thesis, such as a tacnode curve and a plane configuration of three concurrent lines the matrix problem approach is the only method used so far. To describe simple vector bundles on a cuspidal cubic curve and tacnode curve it is sufficient to use matrix reduction in its naive sense. Unfortunately, this is not the case if we consider simple vector bundles on three concurrent lines. This problem is quite involved and requires more general and powerful methods to deal with. Such a tool is given by the representation theory of bocses.

Matrix problems and representations of bocses

The matrix problem technique originates from works of Roiter [Ro60], Nazarova [Naz61], Roiter and Drozd [DR67] on integral representation theory. In 1972 Drozd introduced the notion of a bimodule problem. It was the first but quite fruitful attempt to formalize the matrix problem method. In order to make bimodule problems closed under the matrix reduction procedure (reduction of a fragment of a matrix to the form $\left(\begin{smallmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ by the Gauß Algorithm) Crawley-Boevey introduced bimodules with differentials. On the other hand, Roiter and Kleiner [KR75] worked out the formalism of differential graded categories (DGC), which also allows to categorify matrix reduction. The next step in this direction was a modification of the DGC language to the formalism of bocses (bimodules over categories with coalgebra structure), which provides a clear description of the category of representations (both objects and morphisms). In terms of bocses or DGCs one can formulate the base change lemma (Proposition 6.2.5), which is a theoretical footing for all types of matrix reductions. This consideration was the key point for the proof of Drozd's Tame-Wild dichotomy Theorem [Dro79] (see also [CB88, CB90] for an introductory treatment of this technique).

Remark 1.1.4. Note that the first Brauer-Thrall conjecture proven by Roiter (and independently by Auslander) by pure module-theoretical methods can be alternatively proven in a more conceptual way using the formalism of bocses.

Originally, the method of bocses was invented and applied for purely theoretical purposes, for example to prove the wild-tame dichotomy theorems or semi-continuity results. In this thesis we show the computational power of the representation theory of differential biquivers (Roiter bocses), which allows us to obtain explicit canonical forms describing simple vector bundles on degenerations of elliptic curves.

1.2 Overview of results

The main results obtained in this thesis are the following:

- Characterization of simple vector bundles on Kodaira fibers of type II, III and IV in terms of their geometric invariants, which generalizes Atiyah's classification (see Theorem 1.0.1) to the class of curves of arithmetic genus one.
- Explicit description of universal families of simple vector bundles on Kodaira fibers of type II, III and IV in terms of matrix problems.
- Description of a wide class of bocses which can be wild, but are brick-tame.

Cuspidal cubic curve

In Chapter 3 we describe simple vector bundles and torsion free sheaves on a cuspidal cubic curve. The combinatorics of the answer resembles the case of smooth and nodal Weierstraß curves (see Theorem 1.0.1 and Theorem 2.8.2):

Theorem 1.2.1. *Let E be a cuspidal cubic curve over an algebraically closed field $\Bbbk$. Then*

1. *the rank r and the degree d of a simple torsion free sheaf $\mathcal{F}$ over E are coprime;*

2. *for every pair of coprime integers $(r,d) \in \mathbb{N} \times \mathbb{Z}$, the fine moduli space of simple torsion free sheaves $\mathsf{TF}_E^s(r,d)$ is isomorphic to the curve E itself. Moreover, under this identification vector bundles correspond to regular points $E_{reg} \cong \mathsf{Pic}_E^d \cong \mathbb{A}^1$ and there exists a unique torsion free and not locally free sheaf compactifying the universal family of vector bundles.*

Note that for a cuspidal cubic curve a vector bundle is simple if and only if it is stable. It was already mentioned before that the same result can be also obtained by other methods, for example by using Fourier-Mukai transforms or the GPB approach (see Appendix D and Section 3.7). The main advantage of our technique is that we not only describe moduli spaces of simple torsion free sheaves, but also give explicit algorithms for constructing the universal family of simple vector bundles of given rank and degree. To be precise: by Algorithm 3.2.2 one can construct a canonical form of the matrix corresponding to a vector bundle $\mathcal{E}$ of prescribed rank r, degree d and determinant $\det(\mathcal{E}) \in \mathsf{Pic}_E^d$. Algorithm 3.3.1 does the same for the compactifying torsion free and not locally free sheaf. In Section 3.6 we give an explicit description of a universal bundle of $\mathsf{VB}_E^s(r,d)$.

Thus, putting together Theorem 1.0.1, Theorem 2.8.2 and Theorem 1.2.1 we obtain a uniform description of simple vector bundles on all irreducible degenerations of an elliptic curve. Often such a curves are called a *Weierstraß cubics*: plane cubic curves given by the equation $y^2z = 4x^3 + g_2xz^2 + g_3z^3$, where $(x : y : z)$ are homogeneous coordinates on $\mathbb{P}^2$ and $g_2, g_3 \in \Bbbk$ are constants. A Weierstraß cubic has at most one singular point and is singular if and only if $g_2^3 = 27g_3^2$. Unless $g_2 = g_3 = 0$, the singularity is a node, whereas in the case $g_2 = g_3 = 0$ it is a cusp.

Kodaira fibers of types III and IV

From the point of view of applications in mathematical physics it is important to consider vector bundles on reducible degenerations of elliptic curves. In [DG01, Theorem 2.11] Drozd and Greuel described all indecomposable torsion free sheaves on cycles of projective lines (Kodaira fibers I_N). In [BDG01, Theorem 5.3] a description of simple vector bundles on I_N is deduced from the description of all indecomposable.

In our work we deal with two other plane reducible curves of arithmetic genus one: a tacnode curve (Kodaira fiber of type III, which is a configuration of two projective lines touching at one point) and a configuration of three concurrent projective lines in a plane (Kodaira fiber of type IV).

Let E be either a tacnode curve or three concurrent lines in a plane, $N = 2, 3$ the number of components, $L_k \cong \mathbb{P}^1$ the k-th component of E. For a vector bundle $\mathcal{E}$ on E we denote

- $d_k = d_k(\mathcal{E}) = \deg(\mathcal{E}|_{L_k}) \in \mathbb{Z}$ the degree of the restriction of $\mathcal{E}$ on L_k;
- $\mathbb{d} = \mathbb{d}(\mathcal{E}) = (d_1, \dots, d_N) \in \mathbb{Z}^N$ the *multidegree* of $\mathcal{E}$;
- $d = \deg(\mathcal{E}) = \chi(E) = d_1 + \cdots + d_N$ the degree of $\mathcal{E}$, which coincides with Euler-Poincaré characteristic: $\chi(\mathcal{E}) = h^0(\mathcal{E}) - h^1(\mathcal{E})$;
- $r = rank(\mathcal{E})$ the rank of $\mathcal{E}$.

The following theorem generalizes Atiyah's classification of vector bundles on a smooth elliptic curve and is the main result of this PhD thesis.

Theorem 1.2.2. *Let E be a Kodaira fiber of type II, III or IV. Let $\mathcal{E}$ be a simple vector bundle on E. Then*

$$g.c.d.(r, d) = 1.$$

Moreover, $\mathcal{E}$ is determined by its rank r, multidegree $\mathbb{d}$ and its determinant $\det(\mathcal{E}) \in \mathsf{Pic}^{\mathbb{d}}(E) = \mathbb{A}^1$.

Conversely, for a tuple of integers $(r, \mathbb{d}) \in \mathbb{N} \times \mathbb{Z}^N$ satisfying the condition above there exists a non-empty family of simple vector bundles of rank r and multidegree $\mathbb{d}$ on E parameterized by the points of an affine line $\mathbb{A}^1 \cong \mathsf{Pic}^{\mathbb{d}}(E)$.

Remark 1.2.3. The same description can be obtained for vector bundles on Kodaira cycles I_N $(1 \leq N \leq 3)$. However, we do not prove the result for curves I_2 and I_3 in this thesis.

We prove this result in Chapter 4 for a tacnode curve and in Chapter 5 for three concurrent lines in a plane. The main ingredient of the proof is a construction of various bijections

$$\mathsf{VB}^s_E(r, \mathbb{d}) \longrightarrow \mathsf{VB}^s_E(r', \mathbb{d}'), \tag{1.3}$$

where $r' < r$. This is done using a reduction of our classification problem to a description of bricks in the category of representations of a certain differential biquiver. Moreover, we provide explicit Algorithms 4.4.1 and 5.3.1 which for a given tuple $(r, \mathbb{d}) \in \mathbb{N} \times \mathbb{Z}^N$ construct a canonical form of a matrix describing a universal family of simple vector bundles of rank r and multidegree $\mathbb{d}$. The kernels of these algorithms are automatons of matrix reduction. For Kodaira fibers III and IV we give the brick-reduction automatons ((7.5.1) and (7.6.4)), where paths are brick-reductions, and also the principal reduction automatons (7.5.2) and 7.6.5, which encode reductions (1.3) for bundles.

It is quite plausible that our reduction algorithm can be categorified using Fourier-Mukai transforms, see conjectures in Appendix D. Namely, if we choose for each $1 \leq k \leq N$ a smooth point p_k lying on the component L_k, then the action of the semigroup of automaton (7.5.2) on $\mathbb{Z}^{N+1}$ coincides with the braid group action induced by Seidel-Thomas twists $T_{\mathcal{O}}, T_{p_1}, \dots, T_{p_N} \in \mathrm{Aut}(\mathcal{D}^b(\mathsf{Coh}_E))$ (see [ST01]) on the discrete part of the K–group of the triangulated category of perfect complexes

$$K(\mathsf{Perf}_E) \cong \mathbb{Z} \times \mathsf{Pic}(E) \cong \mathbb{Z}^{N+1} \times \Bbbk.$$

1.3 Organization of the material

Chapter 2. We start by recalling a general construction of Drozd and Greuel, which is the main tool of our method to study vector bundles and torsion free sheaves on degenerations of elliptic curves. The key idea of this approach is a reduction of the description of torsion free sheaves on a rational projective curve to a certain problem of linear algebra. In Section 2.1 we recall the theoretical background concerning this method. Then in Section 2.2 we give an explicit algorithm, used in subsequent chapters, which describes the matrix problem corresponding to the classification of indecomposable torsion free sheaves on a reduced rational projective curve. This chapter also contains general definitions and remarks which we use later on for each concrete curve in question.

In Section 2.5 we describe the matrix problems arising in the case of a nodal cubic curve and a cycle of N projective lines (Kodaira fibers of type I_N). For these curves, the problems are tame and belong to the well-known class of problems of linear algebra called “representations of bunches of chains” [Bon92], see also [CB89] and [KL86].

In Section 2.8 we obtain an alternative proof of the main result of [Bur03] describing simple vector bundles on the nodal cubic curve $zy^2 = x^3 + x^2z$. By the way, it illustrates the so-called *small matrix reduction* considered in [Dro92]. This approach can be used to describe matrices in general position, which in our case are precisely the bricks, i.e. matrices corresponding to simple vector bundles.

Chapters 3, 4 and 5 are devoted to the study of torsion free sheaves on Kodaira fibers of type II, III and IV respectively. In each case it is convenient to separate the problems for vector bundles and torsion free sheaves. Altogether, in this dissertation we deal with five classification problems:

(*i*) vector bundles on a cuspidal cubic curve;

(*ii*) torsion free sheaves on a cuspidal cubic curve;

(*iii*) vector bundles on a tacnode curve;

(*iv*) torsion free sheaves on a tacnode curve;

(*v*) vector bundles on a configuration of three concurrent lines in a plane.

All of them are wild with respect to the classification of indecomposable objects. However, restricted to the subclass of bricks all these problems behave tamely. In each of these cases the whole classification procedure for simple objects can be divided into 4 steps.

- By applying the general method from Section 2.2 we reformulate the classification problem for torsion free sheaves to a matrix problem called MP_E .
- Primary reduction. Since the matrix problem MP_E is quite cumbersome, it is convenient to reduce a brick-object to a partial canonical form and formulate a new matrix problem denoted by BM_P (standing for a "block-matrix category").
- Solve the matrix problem BM_P^s, where index "s" indicates the full subcategory of bricks. This part requires some special techniques and will be treated separately from the rest in Chapter 7. The choice of each matrix reduction step and, consequently, the resulting canonical form is not unique. In order to keep track of the whole picture we introduce the so-called *brick-reduction automaton*. We also present the so-called *principal matrix reduction automaton*. For example, for bundles it describes certain functorial bijections $\mathsf{VB}_E^s(r, \mathbb{d}) \to \mathsf{VB}_E^s(r', \mathbb{d}')$ between sets; if the set $\mathsf{VB}_E^s(r, \mathbb{d}) \cong \mathsf{BM}_P^s(\mathbb{s})$ is nonempty then there is a path p reducing the dimension vector $\mathbb{s}$ to $(1, 0, \ldots, 0)$:

$$\begin{tikzcd}
\mathsf{VB}_E^s(r, \mathbb{d}) \arrow[d, "\cong"] & & \mathsf{Pic}_E^{(0,\ldots,0)} \arrow[d, "\cong"] \\
\mathsf{BM}_P^s(\mathbb{s}) \arrow[rr, "p", "\sim"'] & & \mathsf{BM}_{P'}^s(1, 0, \ldots, 0).
\end{tikzcd}$$

- Present an algorithm based on the principal automaton which recovers a canonical form of matrices of the matrix problem BM_P^s corresponding to the simple vector bundles with prescribed rank and multidegree.

Comparing the reduced matrix problems (*i*)-(*v*) we observe that each new problem is a generalization of the previous one. This seems to be an algebraic shadow for the geometric degeneration of a family of cuspidal curves into a tacnode curve and then into a configuration of three concurrent lines.

Chapter 3. This chapter is devoted to a description of simple torsion free sheaves on a cuspidal cubic curve. In Section 3.1 we reduce the problem (i) to a matrix problem. In Section 3.2 we classify simple vector bundles following the same lines as in Section 2.8 for a nodal cubic curve. The matrix problem describing simple torsion free sheaves is obtained in Section 3.3. We solve this problem in Section 7.2. It is interesting to note that the matrix problem (i) is a degeneration of the analogous problem for a nodal curve (see Section 2.8) and provides an algebraic description of the geometric degeneration of a family of nodal cubic curves to a cuspidal one. In Remark 6.4.6 we write down a precise degeneration in terms of differential biquivers. In Section 3.5 we describe the automaton of the reduction, which turns out to be the same both for nodal and cuspidal cubic curves. In such a way we get a uniform classification of simple torsion free sheaves on irreducible cubic curves.

In Section 3.6 we prove that the canonical form of matrices describing simple vector bundles of given rank r and degree d on a singular irreducible cubic curve E allows one to write down an explicit presentation of a universal family $\mathcal{P}(r,d)$ as a pull-back of two concrete morphisms of coherent sheaves on $E \times \mathsf{VB}^s_E(r,d)$.

In Section 3.7 we give some remarks and compare our construction with the approach of generalized parabolic bundles.

Chapter 4. In this chapter we study simple torsion free sheaves on a tacnode curve. As in the case of a cuspidal cubic curve, we first consider the case of vector bundles. In Section 4.1 we apply the procedure from Section 2.2 to formulate the matrix problem MP_E . In Section 4.2 we perform a primary matrix reduction and formulate the matrix problem BM_P . This problem with respect to bricks is solved in Section 7.3 using the technique of bocses. In Section 4.4, we translate back the results obtained in Section 7.3 in terms of bundles and provide an explicit algorithm to construct the simple bundles with prescribed discrete parameters.

Section 4.5 is devoted to a description of simple torsion free sheaves on a tacnode curve having the same rank on each irreducible component. This treatment is completely parallel to the case of vector bundles: we formulate the matrix problem, apply the primary reduction finally solve it in Section 7.7 using the technique of bocses.

Note that there exist simple torsion free sheaves having different ranks on each irreducible component (take for example the structure sheaf of an irreducible component). The matrix problem for them is also of discrete brick-type as it follows from Theorem 6.7.7 but it will be studied in details elsewhere.

Chapter 5. In this chapter we deal with simple vector bundles on a configuration of three concurrent lines in a plane. The presentation is completely parallel to the case of Kodaira fibers of types II and III: in Section 5.1 we formulate the original matrix problem, in Section 5.2 we make the primary matrix reduction and formulate the BM_P matrix problem and finally solve it in Section 7.6. In Section 5.3 we give an algorithm for constructing canonical forms.

Chapter 6. In order to treat the BM_P problems formally we need the language of bocses. It enables us to think of matrix reduction as some kind of formal calculus.

It turns out that most of the problems of linear algebra arising naturally can be interpreted as categories of representations of normal triangular bocses. In this general framework the class of matrix problems is well defined and closed under the matrix reduction. It seems that many of the basic technical results concerning this theory are not explicitly present in the common literature, so for the reader's convenience we recall some basic notions and facts on bocses and their representations in Chapter 6.

In Section 6.1 we start with some commonly used notions from category theory. Then in Section 6.2 we define bocses and their representations and prove some basic properties of them. Section 6.3 is devoted to the so-called normal free triangular bocses, also called Roiter bocses. Simultaneously we introduce their parallel description in terms of differential biquivers. This one-to-one correspondence is analogous to the correspondence between basic hereditary algebras and quivers.

It turns out that all matrix problems relevant to our classification problems can be reduced to a description of representations of certain Roiter bocses. We give the calculations in Section 6.4, and present a reduction algorithm in Section 6.5. In Section 6.6 we explain the matrix reduction for subcategories of bricks.

The main result of this Chapter is proven in Section 6.7. There we introduce a certain class of Roiter bocses denoted by BT and containing all BM_P problems from Chapters 3-5. The main theorem of this section (Theorem 6.7.7) claims that the class BT is closed under the matrix reductions, and a bocs from BT contains at most one one-parameter family of bricks in each vector dimension. In other words, all bocses from the BT-class are brick-tame in the sense of Definition B.0.9.

Chapter 7. In this chapter we apply the technique developed in previous chapter to BM_P problems from Chapters 3-5. In Sections 7.1-7.3 we treat problems (i)-(iii). In Section 7.4 we describe brick-reductions by an automaton. For reduction on bundles (1.3) (respectively, torsion free sheaves). we introduce the principal reduction automaton. This enables us to calculate the combinatorics of discrete parameter. In Section 7.6 we treat the problem (v), we provide the brick-reduction and principal automatons 7.6.4, 7.6.5 and thus prove Theorem 1.2.2. In Section 7.7 we solve the problem (iv) using calculations from the previous sections.

Appendix A. We consider reduced projective curves of arithmetic genus zero. It is interesting to note that all these curves excluding chains of projective lines are vector-bundle-wild. However, all of them have discrete type with respect to the classification of simple vector bundles.

Appendix B. Here we recall the classical definitions of the tame and wild representation types of a Roiter bocs. Moreover, we introduce the new definitions of brick-tame and brick-wild representation types.

Appendix C. In this appendix we show that any bimodule problem can be rewritten as the category of representations of a certain linear Roiter bocs.

Appendix D. Here we recall the method of Fourier-Mukai transforms on a reduced projective curve E of arithmetic genus one. The most important ingredients of this approach are the so-called twist functors of Seidel and Thomas, also known in representation theory of finite-dimensional algebras under the name "tubular mutations". It is quite plausible that the action of the group $\mathrm{Aut}(\mathcal{D}^b(\mathsf{Coh}_E))$ of exact auto-equivalences of the derived category on the discrete part of the K-group categorifies the action of the semi-group of the brick-reduction automaton on discrete parameters of bricks. We are going to come back to this question in a future work.

Chapter 2

Vector bundles on singular curves via matrix problems

2.1 Category of triples

Let $\Bbbk$ be an algebraically closed field[1] of characteristic zero, let $\mathsf{Sch} := \mathsf{Sch}/_{\Bbbk}$ be the category of Noetherian separated schemes over $\Bbbk$. For any scheme $T \in \mathsf{Sch}$ we denote by VB_T, TF_T and Coh_T the categories of vector bundles, torsion free and coherent sheaves on T respectively.

Let X be a reduced singular projective curve over $\Bbbk$. Introduce the following notation:

- $\pi : \widetilde{X} \longrightarrow X$ the normalization of X;
- $\mathcal{O} := \mathcal{O}_X$ and $\widetilde{\mathcal{O}} := \mathcal{O}_{\widetilde{X}}$ the structure sheaves of X and $\widetilde{X}$ respectively;
- $\mathcal{J} = Ann_{\mathcal{O}}(\pi_*\widetilde{\mathcal{O}}/\mathcal{O})$ the conductor of $\mathcal{O}$ in $\pi_*\widetilde{\mathcal{O}}$;
- $\imath : S \hookrightarrow X$ the subscheme of X defined by the conductor $\mathcal{J}$ and $\tilde{\imath} : \widetilde{S} \hookrightarrow \widetilde{X}$ its scheme-theoretic pull-back to the normalization $\widetilde{X}$.

Altogether they fit into a natural cartesian diagram:

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\tilde{\imath}"] \arrow[d, "\tilde{\pi}"'] & \widetilde{X} \arrow[d, "\pi"] \\ S \arrow[r, "\imath"] & X. \end{tikzcd} \tag{2.1}$$

In the following remark we collect some observations about the objects defined above.

Remark 2.1.1. 1. The main property of the conductor: for $\widetilde{\mathcal{J}} := \pi^*\mathcal{J}/\mathsf{tor}(\pi^*\mathcal{J})$, we have $\mathcal{J} = \pi_*\widetilde{\mathcal{J}}$.

[1] Although the construction of triples and many classification results are valid for an arbitrary field, the matrix problems can be quite special and require different methods to deal with. In order to get a uniform description for all cases we assume from the beginning the ground field $\Bbbk$ to be algebraically closed of characteristic zero.

2. Let $\mathcal{F} \in \mathsf{Coh}_X$ and $\widetilde{\mathcal{F}} \in \mathsf{Coh}_{\widetilde{X}}$ be coherent sheaves on X and $\widetilde{X}$ respectively. With a little abuse of notation one can write: $\imath^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}_S = \mathcal{F}/\mathcal{J}\mathcal{F} \in \mathsf{Coh}_S$ and $\tilde{\imath}^*\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}} \otimes_{\widetilde{\mathcal{O}}} \mathcal{O}_{\widetilde{S}} = \widetilde{\mathcal{F}}/\widetilde{\mathcal{J}}\widetilde{\mathcal{F}} \in \mathsf{Coh}_{\widetilde{S}}$. Since S and $\widetilde{S}$ are schemes of dimension zero, $\imath_*\imath^*\mathcal{F}$ and $\tilde{\imath}_*\tilde{\imath}^*\widetilde{\mathcal{F}}$ are skyscraper sheaves on X and $\widetilde{X}$ respectively.

3. In what follows we shall identify the structure sheaf $\mathcal{O}_T$ of an artinian scheme T with the coordinate ring $\Bbbk[T]$.

4. If X consists of N smooth components L_k, then we write $X = \bigcup\limits_{k=1}^{N} L_k$ and $\widetilde{X} = \bigsqcup\limits_{k=1}^{N} L_k$.

The usual way to deal with vector bundles on a singular curve is to lift them to the normalization, and then to work on a smooth curve, see for example [Ses82, Bho92, Bho96]. Passing to the normalization we lose information about the isomorphism classes of objects of VB_X, since non-isomorphic vector bundles can have isomorphic inverse images. In order to describe the fibers of the map $\mathsf{VB}_X \longrightarrow \mathsf{VB}_{\widetilde{X}}$ and to be able to work with arbitrary torsion free sheaves we introduce the following formalism:

Definition 2.1.2. *The category of triples* Tr_X is defined as follows:

- Its objects are triples $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$, where $\widetilde{\mathcal{F}} \in \mathsf{VB}_{\widetilde{X}}$, $\mathcal{M} \in \mathsf{Coh}_S$ and $\tilde{\mu} : \tilde{\pi}^*\mathcal{M} \to \tilde{\imath}^*\widetilde{\mathcal{F}}$ is an epimorphism of $\mathcal{O}_{\widetilde{S}}$–modules, which induces a monomorphism of $\mathcal{O}$–modules

$$\mu : \imath_*\mathcal{M} \longrightarrow \imath_*\tilde{\pi}_*\tilde{\pi}^*\mathcal{M} \xrightarrow{\tilde{\mu}} \imath_*\tilde{\pi}_*\tilde{\imath}^*\widetilde{\mathcal{F}}. \tag{2.2}$$

- A morphism $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu}) \xrightarrow{(F,f)} (\widetilde{\mathcal{F}}', \mathcal{M}', \tilde{\mu}')$ is given by a pair (F, f), where $F : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}'$ is a morphism in $\mathsf{VB}_{\widetilde{X}}$ and $f : \mathcal{M} \to \mathcal{M}'$ is a morphism in Coh_S, such that the following diagram commutes in $\mathsf{Coh}_{\widetilde{S}}$:

$$\begin{tikzcd} \tilde{\pi}^*\mathcal{M} \arrow[r, "\tilde{\mu}"] \arrow[d, "\tilde{\pi}^* f"'] & \tilde{\imath}^*\widetilde{\mathcal{F}} \arrow[d, "\tilde{\imath}^* F"] \\ \tilde{\pi}^*\mathcal{M}' \arrow[r, "\tilde{\mu}'"] & \tilde{\imath}^*\widetilde{\mathcal{F}}'. \end{tikzcd} \tag{2.3}$$

The main reason to introduce the formalism of triples is the following theorem:

Theorem 2.1.3 ([DG01]). *Let $\Psi : \mathsf{TF}_X \longrightarrow \mathsf{Tr}_X$ be the functor mapping a torsion free sheaf $\mathcal{F}$ to the triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$, where $\widetilde{\mathcal{F}} := \pi^*\mathcal{F}/\mathsf{tor}(\pi^*\mathcal{F})$, $\mathcal{M} := \imath^*\mathcal{F}$ and $\tilde{\mu}$ is the canonical morphism*

$$\tilde{\mu} : \tilde{\pi}^*\imath^*\mathcal{F} \longrightarrow \tilde{\imath}^*\pi^*\mathcal{F} \longrightarrow \tilde{\imath}^*\big(\pi^*\mathcal{F}/\mathsf{tor}(\pi^*\mathcal{F})\big).$$

Then Ψ is an equivalence of categories. Moreover, the category of vector bundles VB_X is equivalent to the full subcategory of Tr_X consisting of those triples $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$, for which $\mathcal{M}$ is a free $\mathcal{O}_S$–module and $\tilde{\mu}$ is an isomorphism.

Sketch of the proof. We construct a quasi-inverse functor $\mathsf{Tr}_X \xrightarrow{\Psi'} \mathsf{TF}_X$ as follows. Let $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu}) \in \mathsf{Tr}_X$ be some triple. Note that we have a canonical projection $\widetilde{\mathcal{F}} \twoheadrightarrow \tilde{\imath}_*\tilde{\imath}^*\widetilde{\mathcal{F}}$, which induces a surjective map

$$\pi_*\widetilde{\mathcal{F}} \twoheadrightarrow \pi_*\tilde{\imath}_*\tilde{\imath}^*\widetilde{\mathcal{F}} \xrightarrow{\cong} \imath_*\tilde{\pi}_*\tilde{\imath}^*\widetilde{\mathcal{F}}.$$

Therefore, the triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ defines a pull-back diagram in Coh_X:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, dashed] \arrow[d, dashed] & \imath_*\mathcal{M} \arrow[d, "\mu"] \\
\pi_*\widetilde{\mathcal{F}} \arrow[r, two heads] & \imath_*\tilde{\pi}_*\tilde{\imath}^*\widetilde{\mathcal{F}}.
\end{tikzcd} \tag{2.4}$$

Since taking the pull-back is a functorial operation, we get a functor $\mathsf{Tr}_X \xrightarrow{\Psi'} \mathsf{Coh}_X$. The map μ is injective, $\mathcal{F} \hookrightarrow \pi_*\widetilde{\mathcal{F}}$ is injective as well, so $\mathcal{F}$ is torsion free. It remains to show that the functors Ψ and Ψ' are quasi-inverse to each other. We refer to [DG01, Burb] for details of the proof.

In other words, Theorem 2.1.3 claims that a torsion free sheaf $\mathcal{F}$ on a singular curve X can be reconstructed from its "normalization" $\pi^*(\mathcal{F})/\mathsf{tor}(\pi^*\mathcal{F})$, its pull-back $\imath^*\mathcal{F}$ on S and the "gluing map"

$$\tilde{\mu} : \tilde{\pi}^*\imath^*\mathcal{F} \longrightarrow \tilde{\imath}^*\big(\pi^*\mathcal{F}/\mathsf{tor}(\pi^*\mathcal{F})\big).$$

In Section 3.6 we show that for singular irreducible curves of arithmetic genus one the formalism of triples provides an explicit construction of a universal family for stable vector bundles and a coarse moduli space for torsion free sheaves.

2.2 General approach

Although the statement of Theorem 2.1.3 holds for arbitrary reduced curves, the method based on this theorem can be efficiently used mainly for rational curves, since in this case the description of vector bundles on the normalization is well understood.

Vector bundles on a projective line

According to the classical result known as the Theorem of Birkhoff-Grothendieck, a vector bundle $\widetilde{\mathcal{F}}$ on a projective line $\mathbb{P}^1$ splits into a direct sum of line bundles, thus

$$\widetilde{\mathcal{F}} \cong \bigoplus_{n\in\mathbb{Z}} \left(\mathcal{O}_{\mathbb{P}^1}(n)\right)^{r_n}. \tag{2.5}$$

Let $(z_0 : z_1)$ be homogeneous coordinates on $\mathbb{P}^1$. Then an endomorphism F of $\widetilde{\mathcal{F}}$ can be written in a matrix form:

$$F = \begin{pmatrix} \ddots & 0 & \dots & 0 & 0 \\ \dots & F_{nn} & \dots & 0 & 0 \\ & \vdots & \ddots & \vdots & \vdots \\ \dots & F_{mn} & \dots & F_{mm} & 0 \\ & \vdots & & \vdots & \ddots \end{pmatrix}, \tag{2.6}$$

where F_{mn} are blocks of size $r_m \times r_n$ with coefficients in the vector space

$$\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1}(m)) \cong \Bbbk[z_0, z_1]_{m-n}, \tag{2.7}$$

since a morphism $\mathcal{O}_{\mathbb{P}^1}(n) \to \mathcal{O}_{\mathbb{P}^1}(m)$ is determined by a homogeneous form $Q(z_0, z_1)$ of degree $m - n$. In particular, the matrix F is lower-block-triangular and the diagonal $r_n \times r_n$ blocks F_{nn} are matrices over $\Bbbk$. The morphism F is an isomorphism if and only if all diagonal blocks F_{nn} are invertible.

Application of Theorem 2.1.3 to the classification of vector bundles and torsion free sheaves on rational curves

Let us describe vector bundles on a rational projective curve X with the normalization $\widetilde{X} = \bigsqcup_{k=1}^{N} L_k$, where all components L_k are projective lines. According to Theorem 2.1.3 it is equivalent to the classification of iso-classes of objects in Tr_X. Note that two triples $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ and $(\widetilde{\mathcal{F}}', \mathcal{M}', \tilde{\mu}')$ are isomorphic only if $\widetilde{\mathcal{F}} \cong \widetilde{\mathcal{F}}'$ and $\mathcal{M} \cong \mathcal{M}'$. Therefore we fix representatives from iso-classes of $\widetilde{\mathcal{F}}$ and $\mathcal{M}$ and describe equivalence classes of maps $\tilde{\mu} : \tilde{\pi}^*\mathcal{M} \to \tilde{\imath}^*\widetilde{\mathcal{F}}$ with respect to the action of invertible morphisms

$$(F, f) : \tilde{\mu} \mapsto (\tilde{\imath}^* F) \circ \tilde{\mu} \circ (\tilde{\pi}^* f^{-1}).$$

To be precise, we proceed in several steps as follows:

1. Choose homogeneous coordinates $(z_0 : z_1)$ on each component $L := L_k \cong \mathbb{P}^1$, for $1 \le k \le N$. (We omit the index k when it is clear which component is meant.)

2. Fix the iso-class of the restriction of $\widetilde{\mathcal{F}}$ to each component $L := L_k$:

$$\widetilde{\mathcal{F}}|_L \xrightarrow{\cong} \Big(\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_L(n)\Big)^{r(n,k)} \text{ with } \sum_{n\in\mathbb{Z}} r(n,k) = \operatorname{rank}(\widetilde{\mathcal{F}}) =: r.$$

3. Choose trivializing isomorphisms

$$\tau_{n,k} : \tilde{\imath}^*\mathcal{O}_L(n) \xrightarrow{\cong} \mathcal{O}_{\widetilde{S}\cap L},$$

where $\widetilde{S}\cap L$ denotes the scheme-theoretic preimage of the singular locus S under the map $\pi_L : L \hookrightarrow \widetilde{X} \xrightarrow{\pi} X$. The set of trivializations $\{\tau_{n,k} \mid 1 \leq k \leq N, n \in \mathbb{Z}\}$ induces isomorphisms:

$$\tau_k : \tilde{\imath}^*\widetilde{\mathcal{F}}|_{L\cap\widetilde{S}} \xrightarrow{\cong} \mathcal{O}^r_{L\cap\widetilde{S}}$$

and an isomorphism

$$\tau = (\tau_1, \ldots, \tau_N) : \tilde{\imath}^*\widetilde{\mathcal{F}} \xrightarrow{\cong} \mathcal{O}^r_{\widetilde{S}}.$$

4. For an endomorphism $F \in \operatorname{End}_{\widetilde{X}}(\widetilde{\mathcal{F}})$ we have $F = (F_1, \ldots, F_N)$, where each $F_k : \widetilde{\mathcal{F}}|_{L_k} \to \widetilde{\mathcal{F}}|_{L_k}$ is a matrix with block structure (2.6) and homogenous forms as entries.

   Let $Q \in \operatorname{Hom}_L(\mathcal{O}_L(n), \mathcal{O}_L(m)) \cong \Bbbk[z_0, z_1]_{m-n}$ be a homogeneous form of degree $m - n$. Then there is an induced morphism of $\mathcal{O}_{\widetilde{S}\cap L}$-modules $\tilde{\imath}^*Q : \tilde{\imath}^*\mathcal{O}_L(n) \longrightarrow \tilde{\imath}^*\mathcal{O}_L(m)$ and the morphism

$$\mathcal{O}_{\widetilde{S}\cap L} \xrightarrow{{\tau_{n,k}}^{-1}} \tilde{\imath}^*\mathcal{O}_L(n) \xrightarrow{\tilde{\imath}^*Q} \tilde{\imath}^*\mathcal{O}_L(m) \xrightarrow{\tau_{m,k}} \mathcal{O}_{\widetilde{S}\cap L}\,.$$

   We shall frequently identify this map with $\tilde{\imath}^*Q$. Then each matrix $\tilde{\imath}^*F_k$ of the morphism $\tilde{\imath}^*F = (\tilde{\imath}^*F_1, \ldots, \tilde{\imath}^*F_N)$, has the matrix form (2.6) but with entries $\tilde{\imath}^*Q \in \Bbbk[\widetilde{S}\cap L]$ replacing homogeneous forms Q.

5. If the module $\mathcal{M}$ is free (the case of vector bundles), we take an arbitrary isomorphism $\mathcal{M} \xrightarrow{\cong} \mathcal{O}^r_S$ and have $\operatorname{End}_S(\mathcal{M}) \cong \operatorname{Mat}_{\Bbbk[S]}(r \times r)$ and $\operatorname{Aut}_S(\mathcal{M}) \cong \operatorname{GL}(\Bbbk[S], r)$. For an arbitrary $\mathcal{M} \in \mathsf{Coh}_S$ the choice of a basis is more subtle, we describe it in step 8.

6. If $\tilde{\mu} : \tilde{\pi}^*\mathcal{M} \to \tilde{\imath}^*\widetilde{\mathcal{F}}$ is an invertible morphism of two free modules, it can be presented as a matrix $\tilde{\mu} \in \operatorname{GL}(\Bbbk[\widetilde{S}], r)$.

7. Hence, we obtain the problem to describe the equivalence classes of $\tilde{\mu}$ with respect to the action of the group $\tilde{\imath}^*\big(\operatorname{Aut}(\widetilde{\mathcal{F}})\big) \times \mathrm{GL}(\Bbbk[S], r)$:
$$(\tilde{\imath}^* F, \tilde{\pi}^* f) : \tilde{\mu} \mapsto (\tilde{\imath}^* F) \circ \tilde{\mu} \circ (\tilde{\pi}^* f)^{-1}.$$
Problems like this are called *matrix problems* and invertible morphisms are called *matrix transformations.* For the sake of convenience we choose $\Bbbk$-bases of $\mathcal{O}_S$ and $\mathcal{O}_{\widetilde{S}}$ and rewrite $\tilde{\mu}, \tilde{\imath}^* F$ and $\tilde{\pi}^* f$ as tuples of matrices over $\Bbbk$.

8. In the case of torsion free sheaves we should consider $\Bbbk$-bases of $\tilde{\imath}^*\widetilde{\mathcal{F}}$ and $\tilde{\pi}^*\mathcal{M}$ from the very beginning. Obviously, in general such a basis of $\mathcal{M}$ is not well defined simply because the ring $\mathcal{O}_S$ can have infinitely many non-isomorphic indecomposable modules. However, if there are only finitely many $\mathcal{O}_S$-modules R_j, $1 \le j \le T$, which are restrictions of indecomposable torsion free modules over $\mathcal{O}_{X,s}$ $s \in S$, then we can consider the decomposition $\mathcal{M} \cong \oplus_{j=1}^{T} R_j^{r_j}$, where $\sum\limits_{j=1}^{T} r_j = r$. Fixing a $\Bbbk$-basis of each R_j induces $\Bbbk$-basis of $R_j^{r_j}$ and $\mathcal{M}$. Note that one-dimensional rings with finitely many indecomposable torsion free modules were studied in [Jac67] and [DR67]. In [GK85] one can find a list of reduced curve singularities (X, s) with finitely many indecomposable torsion free $\mathcal{O}_{X,s}$-modules.

Remark 2.2.1. In item 3 the trivialization $\tau_{n,k}$ can be chosen arbitrarily for each component L_k and each twist $n \in \mathbb{Z}$. However, it is natural to demand for the total set of trivializations to be compatible with the tensor product. That means that for any component $L := L_k$ and a set of trivializations $\{\tau_n := \tau_{n,k} | n \in \mathbb{Z}\}$ the following diagram must commute:

$$\begin{tikzcd}
\tilde{\imath}^*\big(\mathcal{O}_L(m) \otimes \mathcal{O}_L(n)\big) \arrow[r, "mult"] \arrow[d] & \tilde{\imath}^*\mathcal{O}_L(m+n) \arrow[dd, "\tau_{m+n}"] \\
\tilde{\imath}^*\mathcal{O}_L(m) \otimes \tilde{\imath}^*\mathcal{O}_L(n) \arrow[d, "\tau_m \otimes \tau_n"'] & \\
\mathcal{O}_{L\cap\widetilde{S}} \otimes \mathcal{O}_{L\cap\widetilde{S}} \arrow[r, "mult"] & \mathcal{O}_{L\cap\widetilde{S}}
\end{tikzcd}$$

For concrete curves, there are natural choices of such trivializations.

Remark 2.2.2. Let us mention that in this work we consider neither tensor products of vector bundles nor tensor products of triples. Our main goal is to describe simple vector bundles on curves of arithmetic genus one, where all vector bundles cannot be classified. On such curves the tensor product of two simple vector bundles is not simple unless at least one of them is a line bundle.

2.3 Matrix problem MP_X

It seems reasonable to categorify the obtained problem.

Let $\mathsf{C} := \mathsf{Mor}_{\widetilde{S}-S}(\tilde{\imath}^*\mathcal{M}, \tilde{\pi}^*\widetilde{\mathcal{F}})$ be the category of morphisms from $\tilde{\imath}^*\mathcal{M}$ to $\tilde{\pi}^*\widetilde{\mathcal{F}}$ compatible with $\mathcal{O}_S \hookrightarrow \tilde{\pi}_*\mathcal{O}_{\widetilde{S}}$. Introduce the *matrix problem category* $\mathsf{MP}_X := \mathsf{Mat}(\mathsf{C})$. The procedure described in steps 1-7 induces a full and dense functor:

$$\mathsf{H} : \mathsf{Tr}_X \longrightarrow \mathsf{MP}_X\,.$$

There is a natural projection

$$\mathrm{Hom}_{\mathsf{Tr}_X}\big((\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu}), (\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu}')\big) \twoheadrightarrow \mathrm{Hom}_{\mathsf{MP}_X}(\tilde{\mu}, \tilde{\mu}'). \tag{2.8}$$

It turns out that the category MP_X is a Krull-Schmidt category and it splits into strata:

$$\mathsf{MP}_X \cong \bigcup_{\mathtt{r}} \mathsf{MP}_X(\mathtt{r}),$$

where each category $\mathsf{MP}_X(\mathtt{r})$ is defined as follows:

- objects of $\mathsf{MP}_X(\mathtt{r})$ are matrices $\tilde{\mu}$ for which there exists a triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu}) \in \mathsf{Tr}_X$ and according to step 2 the vector bundle $\widetilde{\mathcal{F}} \in \mathsf{VB}_{\widetilde{X}}$ splits into a direct sum of line bundles with multiplicities $\mathtt{r} := \{r(n,k) | n \in \mathbb{Z},\ 1 \le k \le N\}$;

- for a pair of objects $\tilde{\mu}, \tilde{\mu}' \in \mathsf{MP}_X(\mathtt{r})$

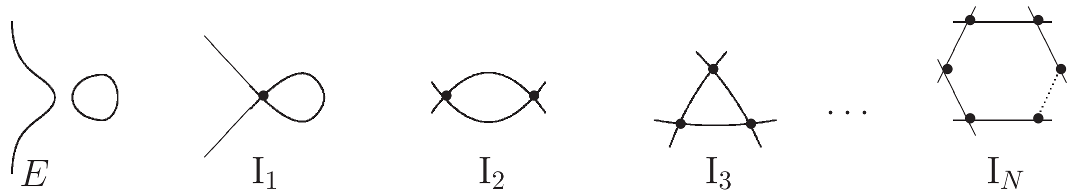
  $$\mathrm{Hom}_{\mathsf{MP}_X}(\tilde{\mu}, \tilde{\mu}') := \{(\tilde{\imath}^*F, \tilde{\pi}^*f) |\ \tilde{\imath}^*F \circ \tilde{\mu} = \tilde{\mu}' \circ \tilde{\pi}^*f\},$$

  where $F \in \mathrm{End}(\widetilde{\mathcal{F}})$ and $f \in \mathrm{End}(\mathcal{M})$.

Definition 2.3.1. Invertible morphisms in $\mathsf{MP}_X(\mathtt{r})$ are called *matrix transformations*. Replacing the set of morphisms by the set of matrix transformations we obtain a *groupoid* assigned to the stratum category $\mathsf{MP}_X(\mathtt{r})$. A *matrix problem* is the problem of describing orbits of indecomposable (respectively simple) objects of this groupoid. If it is possible, a solution consists in finding a *canonical form* of $\tilde{\mu}$.

The matrix problem approach, that we are going to use, was introduced by Kiev Representation theory school. It is quite helpful for various classification problems arising in different fields of mathematics. The key result of this method is Drozd's Tame-Wild Theorem B.0.11, saying that any matrix problem of infinite type is either tame or wild. Roughly speaking, "a problem is tame" means that indecomposable objects can be classified by finite set of canonical forms for each vector dimension, and "wild" means the problem contains all representations of all finitely generated algebras, i.e. indecomposable objects cannot be classified. A classical wild problem is the problem of describing indecomposable representations of $\Bbbk\langle x, y\rangle$. A curve with a wild (tame) category of vector bundles is called *vector-bundle-wild,* or simply *wild* (respectively *vector-bundle-tame* or simply *tame*).

In [DG01] it was shown that all curves of arithmetic genus > 0 are wild with the exceptions of elliptic curves and cycles of projective lines (Kodaira fibers I_N).



Vector bundles over an elliptic curve were described using a different method by Atiyah in [Ati57]. Indecomposable vector bundles and coherent sheaves on cycles of projective lines were classified in [DG01], [BD04] and [Burb]. In this work we study vector-bundle-wild curves whose simple vector bundles, however, behave tamely (see Appendix B for precise definitions). Although Kodaira cycles I_N and Kodaira fibres of types II, III and IV lead to matrix problems of different representation types, they share similar properties concerning simple vector bundles. In order to be able to illustrate this resemblance we recall the matrix problem for vector bundles on cycles of projective lines. In both cases after describing matrices $\tilde{\mu}$ by their canonical forms we interpret the result in terms of sheaves.

2.4 Riemann-Roch theorem

In this section we recall the relationship between the ranks r_k from equation (2.5) and the rank r, the degree d and the multidegree $\mathbb{d}$.

For a projective curve X and $\mathcal{F} \in \mathsf{Coh}_X$ we write $h^i(\mathcal{F}) := \dim_{\Bbbk} H^i(\mathcal{F})$ and $\chi(\mathcal{F}) := h^0(\mathcal{F}) - h^1(\mathcal{F})$ the Euler-Poincaré characteristic.

Let E be a singular reduced projective curve of arithmetic genus one with N components and normalization $\widetilde{E} = \bigsqcup\limits_{k=1}^{N} L_k$, where all components L_k are projective lines. Let $\mathcal{F}$ be a torsion free sheaf of rank r with the corresponding triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$. Define the degree of $\mathcal{F}$ by $\deg_E(\mathcal{F}) := \chi(\mathcal{F})$. Recall that by the Riemann-Roch theorem

$$\deg_{\widetilde{E}}(\widetilde{\mathcal{F}}) = \chi(\widetilde{\mathcal{F}}) - r\chi(\mathcal{O}_{\widetilde{E}}) = \chi(\widetilde{\mathcal{F}}) - rN = \sum_{k=1}^{N} \sum_{n_k \in \mathbb{Z}} n_k r_{n_k},$$

hence,

$$\deg_E(\pi_*\widetilde{\mathcal{F}}) = \chi(\widetilde{\mathcal{F}}) = rN + \sum_{k=1}^{N} \sum_{n_k \in \mathbb{Z}} n_k r_{n_k}.$$

Considering the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J}\mathcal{F} \arrow[r] \arrow[d, "\cong"] & \mathcal{F} \arrow[r] \arrow[d, "\phi"] & i_*\mathcal{M} \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & \mathcal{J}\pi_*\widetilde{\mathcal{F}} \arrow[r] & \pi_*\widetilde{\mathcal{F}} \arrow[r] & \imath_*\tilde{\pi}_*\tilde{\imath}^*\widetilde{\mathcal{F}} \arrow[r] & 0,
\end{tikzcd} \tag{2.9}$$

from the Snake Lemma we obtain $\operatorname{coker}(\phi) = \operatorname{coker}(\mu)$. Hence, there is an exact sequence:

$$0 \to \mathcal{F} \to \pi_*\widetilde{\mathcal{F}} \to \imath_*(\tilde{\pi}_*\tilde{\imath}^*\widetilde{\mathcal{F}}/\mathcal{M}) \to 0 \tag{2.10}$$

which implies:

$$\begin{aligned} \deg_E(\mathcal{F}) &= \deg_E(\pi_*\widetilde{\mathcal{F}}) - h^0(\tilde{\imath}^*\widetilde{\mathcal{F}}) + h^0(\mathcal{M}) \\ &= \deg_{\widetilde{E}}(\widetilde{\mathcal{F}}) + rN - h^0(\tilde{\imath}^*\widetilde{\mathcal{F}}) + h^0(\mathcal{M}). \end{aligned} \tag{2.11}$$

For each curve E as above one can check that $h^0(\mathcal{O}_{\widetilde{S}}) = 2N$ and hence,

$$h^0(\tilde{\imath}^*\widetilde{\mathcal{F}}) = rh^0(\mathcal{O}_{\widetilde{S}}) = 2rN.$$

In particular, if $\mathcal{E}$ is a vector bundle, we get $h^0(\mathcal{M}) = rh^0(\mathcal{O}_S) = rN$ and

$$\deg_E(\mathcal{E}) = \deg_{\widetilde{E}}(\widetilde{\mathcal{E}}). \tag{2.12}$$

For a torsion free and not locally free sheaf $\mathcal{F}$ having the same rank on each component, we have $h^0(\mathcal{M}) > rN$. If we assume $h^0(\mathcal{M}) = Nr + t$, then

$$\deg_E(\mathcal{F}) = \deg_{\widetilde{E}}(\widetilde{\mathcal{F}}) + t. \tag{2.13}$$

For curves with many components it seems reasonable to introduce some extra geometric invariants:

Definition 2.4.1. Let X be a rational curve with N components such that $\widetilde{X} = \cup_{k=1}^N L_k$. The *multidegree* of a torsion free sheaf $\mathcal{F}$ is the tuple

$$\mathbb{d} := \underline{\deg}(\mathcal{F}) = (d_1, \ldots, d_N), \text{ where } d_k := \deg_{L_k}(\widetilde{\mathcal{F}}|_{L_k}) \text{ for } 1 \le k \le N,$$

where as usual $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}/\mathsf{tor}(\pi^*\mathcal{F})$.

If $\mathcal{E}$ is a vector bundle, then

$$d := \deg_E(\mathcal{E}) = \sum_{k=1}^{N} d_i. \tag{2.14}$$

For a torsion free and not locally free sheaf $\mathcal{F}$ having the same rank on each component, it holds

$$d := \deg_E(\mathcal{F}) = \sum_{k=1}^{N} d_i + t\,. \tag{2.15}$$

Using these formulas in each chapter we reformulate the description of torsion free sheaves in terms of their rank and multidegree.

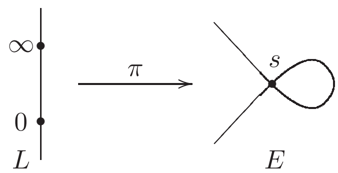
In Appendix A we recall the matrix problems for curves of arithmetic genus 0. In the following section we consider the matrix problem for cycles of projective lines.

2.5 Vector bundles and torsion free sheaves on cycles of projective lines

In this section we apply the method of triples to *cycles of projective lines* (Kodaira fibers of type I_N) and in particular to a nodal cubic curve. According to Proposition 2.7 of [DG01] cycles of projective lines and chains of projective lines form a unique class of singular vector-bundle-tame curves.

Nodal cubic curve

Let E be a nodal cubic curve, given by the equation $zy^2 - x^3 - zx^2 = 0$, let $s = (0:0:1)$ be its singular point and $\mathbb{P}^1 = L \xrightarrow{\pi} E$ be the normalization map.



Choose coordinates on L in such a way that the preimages of s are $0 := (0:1)$ and $\infty := (1:0)$. Then $\mathcal{O}_S = \Bbbk(s)$ and $\mathcal{O}_{\widetilde{S}} = \Bbbk(0) \times \Bbbk(\infty)$.

To describe torsion free shaves on E, for a triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ we fix:

- a decomposition $\widetilde{\mathcal{F}} \cong \bigoplus_{n\in\mathbb{Z}} \widetilde{\mathcal{O}}(n)^{r_n}$, where $\sum_{n\in\mathbb{Z}} r_n = r$;
- an isomorphism $\mathcal{M} \cong \big(\Bbbk(s)\big)^{r+t}$, where $t = 0$ in the case of vector bundles;

- since the choice of coordinates on L fixes two canonical sections z_0 and z_1 of $H^0\big(\widetilde{\mathcal{O}}(1)\big)$, we use the following trivializations

$$\begin{aligned} \widetilde{\mathcal{O}}(n) \otimes \widetilde{\mathcal{O}}/\widetilde{\mathcal{J}} &\xrightarrow{\sim} \Bbbk(0) \times \Bbbk(\infty) \\ \zeta \otimes 1 &\longmapsto (\zeta/z_1^n(0), \zeta/z_0^n(\infty)); \end{aligned}$$

  Note that this isomorphism only depends on the choice of coordinates of $\mathbb{P}^1$. In such a way we equip the $\mathcal{O}_{\widetilde{S}}$–module $\tilde{\imath}^*\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}(0) \oplus \widetilde{\mathcal{F}}(\infty)$ with a basis and get isomorphisms $\widetilde{\mathcal{F}}(0) \cong \bigoplus\limits_{n\in\mathbb{Z}} \Bbbk(0)^{r_n}$ and $\widetilde{\mathcal{F}}(\infty) \cong \bigoplus\limits_{n\in\mathbb{Z}} \Bbbk(\infty)^{r_n}$.

With respect to all these choices the maps $\tilde{\mu}$, $\tilde{\imath}^*F$ and $\tilde{\pi}^*f$ can be written as matrices.

- A morphism $\tilde{\mu} : \tilde{\pi}^*\mathcal{M} \longrightarrow \tilde{\imath}^*\widetilde{\mathcal{F}}$ can be viewed as a pair $\tilde{\mu} = (\mu(0), \mu(\infty))$ of linear maps of $\Bbbk$–vector spaces. From the definition of the category of triples it follows that the matrices $\mu(0)$ and $\mu(\infty)$ have to be of full row rank and the transposed matrix $(\mu(0)|\mu(\infty))^T$ has to be monomorphic. Vector bundles on E correspond to pairs of invertible square matrices $(\mu(0), \mu(\infty))$.

- If we have a morphism $\widetilde{\mathcal{O}}(n) \to \widetilde{\mathcal{O}}(m)$ given by a homogeneous form $Q(z_0, z_1)$ of degree $m-n$, then it induces a map $\widetilde{\mathcal{O}}(n)\otimes\mathcal{O}_{\widetilde{S}} \longrightarrow \widetilde{\mathcal{O}}(m)\otimes\mathcal{O}_{\widetilde{S}}$ given by $(Q(0:1), Q(1:0)) =: (Q(0), Q(\infty))$. Hence, with respect to the chosen trivializations of $\widetilde{\mathcal{O}}(n)$ at 0 and ∞ the map

$$\tilde{\imath}^*F = \big(F(0), F(\infty)\big) : \Bbbk^r(0) \oplus \Bbbk^r(\infty) \longrightarrow \Bbbk^r(0) \oplus \Bbbk^r(\infty) \tag{2.16}$$

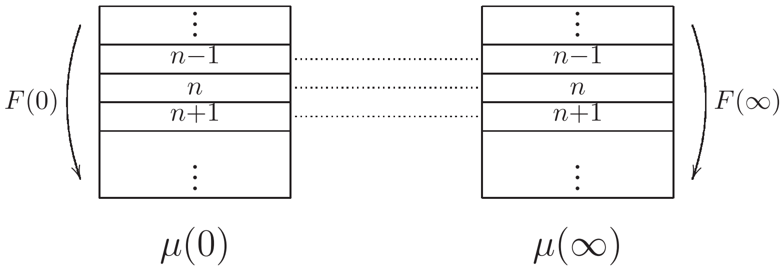
  is given by a pair of lower block triangular matrices $\big(F(0), F(\infty)\big)$ consisting of blocks $F_{mn}(0), F_{mn}(\infty) \in \mathrm{Mat}_{\Bbbk}(r_m \times r_n)$, for $m > n$ and with common diagonal blocks $F_{nn} \in \mathrm{Mat}_{\Bbbk}(r_n \times r_n)$. The morphism F is invertible, if and only if diagonal blocks F_{nn} belong to $\mathrm{GL}(\Bbbk, r_n)$.

- The induced map $\tilde{\pi}^*f = (f, f)$ belongs to the diagonal of the product of $\mathrm{Mat}_{\Bbbk}(r \times r) \times \mathrm{Mat}_{\Bbbk}(r \times r)$. Obviously, if (F, f) is invertible then $f \in \mathrm{GL}(\Bbbk, r)$.

We obtain the following matrix transformations:

$$\big(\mu(0), \mu(\infty)\big) \mapsto \big(F(0)\mu(0)f^{-1}, F(\infty)\mu(\infty)f^{-1}\big), \tag{2.17}$$

where F is an automorphism of $\bigoplus\limits_{n\in\mathbb{Z}} \widetilde{\mathcal{O}}(n)^{r_n}$ and f is an automorphism of $\Bbbk^r$. Note that the blocks $F_{mn}(0)$ and $F_{mn}(\infty)$, $\quad m, n \in \mathbb{Z}, \quad m > n$ are *arbitrary* and F_{nn} *arbitrary invertible* for $n \in \mathbb{Z}$. As a result we get the following *matrix problem.*



Matrix problem for a nodal cubic curve

We have two matrices $\mu(0)$ and $\mu(\infty)$ of the same size and both of full row rank. Each of them is divided into horizontal blocks labeled by integers (they are called sometimes weights). Blocks of $\mu(0)$ and $\mu(\infty)$, labeled by the same integer, have the same size. We are allowed to perform the following transformations:

1. An arbitrary elementary transformation of columns simultaneously for the matrices $\mu(0)$ and $\mu(\infty)$. Such transformation corresponds to the matrix f.

2. An arbitrary invertible elementary transformations of rows simultaneously inside of any two conjugated horizontal blocks of the matrices $\mu(0)$ and $\mu(\infty)$. Such a transformation corresponds to the diagonal blocks of the matrix F.

3. For each of the matrices $\mu(0)$ and $\mu(\infty)$ we can independently add a scalar multiple of any row with a lower weight to any row with a higher weight. Such a transformation corresponds to the non-diagonal blocks of the matrices $F(0)$ and $F(\infty)$.

The main idea behind the matrix reduction is that we can transform the matrix μ into a canonical form which is quite analogous to the Jordan normal form.

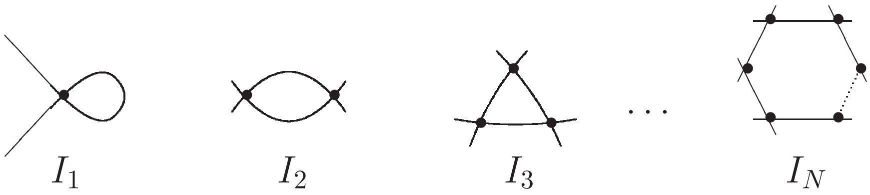
Example 2.5.1. Let E be a nodal cubic curve.

- The following triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ defines an indecomposable vector bundle of rank 2 and degree 1 on E: the normalization $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}} \oplus \widetilde{\mathcal{O}}(1)$, $\mathcal{M} = \Bbbk^2(s)$ and matrices:

$$\mu(0) = \begin{array}{c} \boxed{\begin{array}{cc} 1 & 0 \\ \hline 0 & 1 \end{array}} \end{array}\begin{array}{l} {\scriptstyle 0} \\ {\scriptstyle 1} \end{array} \quad \text{and} \quad \mu(\infty) = \begin{array}{c} \boxed{\begin{array}{cc} 0 & 1 \\ \hline \lambda & 0 \end{array}} \end{array}\begin{array}{l} {\scriptstyle 0} \\ {\scriptstyle 1} \end{array} \qquad \lambda \in \Bbbk^*.$$

- The triple $\big(\widetilde{\mathcal{O}}(-1), \Bbbk^2, \tilde{\mu} = \boxed{1\ \ 0}\ \boxed{0\ \ 1}\big)$ describes the unique torsion free but not locally free sheaf of degree zero, which compactifies the Jacobian $\mathrm{Pic}^0(E)$.

Cycles of projective lines (Kodaira cycles)



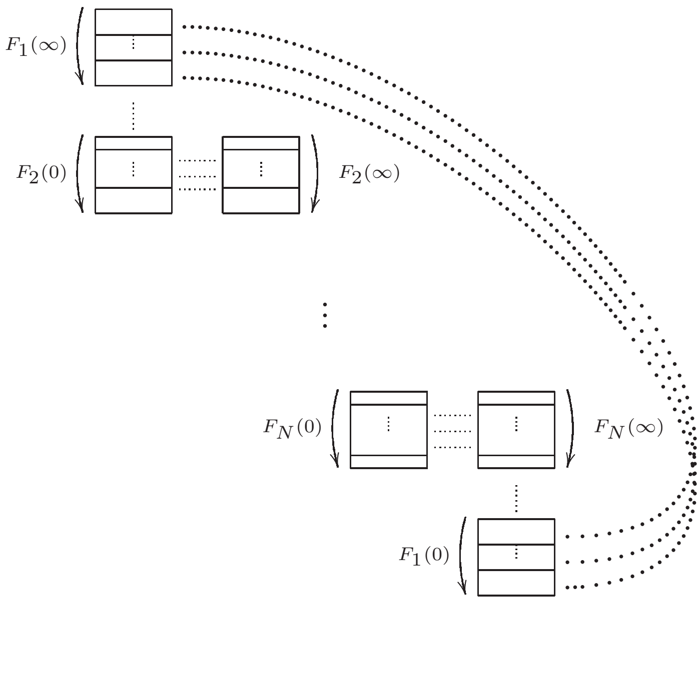
For a cycle of N projective lines we obtain a similar problem to the case of a nodal cubic curve. This time, however the gluing map $\tilde{\mu} : \tilde{\pi}^*\mathcal{M} \to \tilde{\imath}^*\widetilde{\mathcal{F}}$ is given by $2N$ matrices

$$\tilde{\mu} = \big(\mu_1(0), \mu_1(\infty), \mu_2(0), \mu_2(\infty), \ldots, \mu_N(0), \mu_N(\infty)\big)$$

and transformation rule is

$$\begin{cases} \mu_1(0) & \mapsto F_1(0)\mu_1(0)f_N^{-1}, \\ \mu_1(\infty) & \mapsto F_1(\infty)\mu_1(\infty)f_1^{-1}, \\ \mu_2(0) & \mapsto F_2(0)\mu_2(0)f_1^{-1}, \\ \mu_2(\infty) & \mapsto F_2(\infty)\mu_2(\infty)f_2^{-1}, \\ \vdots & \vdots \\ \mu_{N-1}(0) & \mapsto F_{N-1}(0)\mu_{N-1}(0)f_{N-2}^{-1}, \\ \mu_{N-1}(\infty) & \mapsto F_{N-1}(\infty)\mu_{N-1}(\infty)f_{N-1}^{-1}, \\ \mu_N(0) & \mapsto F_N(0)\mu_N(0)f_{N-1}^{-1}, \\ \mu_N(\infty) & \mapsto F_N(\infty)\mu_N(\infty)f_N^{-1}. \end{cases} \tag{2.18}$$

It can be sketched as follows:



We are allowed to perform the following transformations.

1. An arbitrary elementary invertible transformation of columns simultaneously for the matrices $\mu_k(\infty)$ and $\mu_{k+1}(0)$ for $1 \le k \le N$, where $\mu_{N+1} := \mu_1$. Such transformations correspond to the matrices f_k.
2. An arbitrary elementary invertible transformation of rows simultaneously inside of any two conjugated horizontal blocks of the matrices $\mu_k(0)$ and $\mu_k(\infty)$. Such transformations correspond to the diagonal blocks of F_k.
3. For each of the matrices $\mu_k(0)$ and $\mu_k(\infty)$ we can independently add a scalar multiple of any row with a lower weight to any row with a higher weight. Such transformations correspond to non-diagonal blocks of $F_k(0)$ and $F_k(\infty)$.

These types of matrix problems are well-known in representation theory. First they appeared in the work of Nazarova and Roiter [NR69] about the classification of $\mathbb{k}[\![x, y]\!]/(xy)$–modules. They are called, sometimes, "Gelfand problems" or "representations of bunches of chains" (for example, see [Bon92]). For application of "Gelfand problems" to classification of vector bundles and torsion free sheaves on cycles of projective lines we refer to [DG01] (see also[BBDG]). A description of torsion free sheaves on cycles of projective lines is given in Theorem 1.1.3 in Introduction and we do not repeat it here.

2.6 Simplicity condition

As it was mentioned above, the main goal of this work is to give a classification of *simple* torsion free sheaves and vector bundles on a projective curve of arithmetic genus one.

Definition 2.6.1. A torsion free sheaf is called *simple* if it admits no endomorphisms but homotheties, i.e. $\mathrm{End}_X(\mathcal{F}) = \mathbb{k}$. The subcategory of simple torsion free sheaves and the subcategory of simple vector bundles are denoted by TF^s_X and VB^s_X respectively.

The notion of simplicity naturally translates to the language of categories MP_X introduced in Section 2.3. We say that an object $\tilde{\mu}$ of MP_X is *simple* (or a *brick*) if it admits no nontrivial endomorphisms. The full subcategory of simple objects is denoted by MP^s_X and $\mathsf{MP}^s_X(\mathbb{r})$ if the tuple of sizes $\mathbb{r}$ is fixed.

However, computing $\mathrm{End}_X(\mathcal{F})$ we should take into account not only restricted pairs $(\tilde{\imath}^*F, \tilde{\pi}^*f) \in \mathrm{End}_{\mathsf{MP}_X}(\tilde{\mu})$ but also their preimages (F, f) in the category of triples Tr_X . It can happen that a nonscalar morphism (F, f) has a scalar restriction $(\tilde{\imath}^*F, \pi^*f)$. To be precise the following simple lemma holds:

Lemma 2.6.2. *Following the notations of Section 2.2 let X be a singular rational projective curve and $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu}) \in \mathsf{Tr}_X$ be a triple. Then the map (2.8): $\mathrm{End}_{\mathsf{Tr}_X}(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu}) \longrightarrow \mathrm{End}_{\mathsf{MP}_X}(\tilde{\mu})$ is bijective if and only if for all components L of $\widetilde{X}$ and for all summands $\mathcal{O}_L(n) \oplus \mathcal{O}_L(m)$ of $\widetilde{\mathcal{F}}|_L$ the canonical maps $\mathrm{Hom}(\mathcal{O}_L(n), \mathcal{O}_L(m)) \to \Bbbk[\widetilde{S} \cap L]$, taking $Q \mapsto \tilde{\imath}^* Q$ are bijective.*

Proof. The statement follows immediately from the definition of the category MP_X. □

This lemma implies certain nice properties for a matrix problem under the simplicity condition. For instance, we have the following:

Lemma 2.6.3. *Let X be a singular curve, $L \cong \mathbb{P}^1$ be a rational component of the normalization such that $L \cap \widetilde{S} = \{s_1, s_2\}$, where s_1 and s_2 are two different points, and let $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu}) \in \mathsf{Tr}_X$ be a simple triple. Then $\widetilde{\mathcal{F}}|_L = \mathcal{O}_L(n)^{r_1} \oplus \mathcal{O}_L(n+1)^{r_2}$, for some $n \in \mathbb{Z}$ and non-negative integers r_1, r_2 such that $r_1 + r_2 = r := \mathrm{rank}(\widetilde{\mathcal{F}})$.*

Proof. Assume $\pi^*\mathcal{F}|_L$ contains a summand $\mathcal{O}_L(n) \oplus \mathcal{O}_L(m)$ with $m > n+1$. Choose homogeneous coordinates $(z_0 : z_1)$ on L such that the points s_1 and s_2 have coordinates $0 := (0:1)$ and $\infty := (1:0)$. Since the degree $m - n \geq 2$ there exists a nonzero homogeneous form $Q \in \mathrm{Hom}_L(\mathcal{O}_L(n), \mathcal{O}_L(m)) \cong \Bbbk[z_0, z_1]_{m-n}$ such that $\tilde{\imath}^* Q = (Q(0), Q(\infty)) = 0$. Thus the map $Q \mapsto \tilde{\imath}^* Q$ is not injective and we get a contradiction to the conditions of Lemma 2.6.2. □

Lemma 2.6.4. *Let X be a singular curve with a rational component $L \cong \mathbb{P}^1$ such that the restriction of $\widetilde{\mathcal{J}} = \pi^*\mathcal{J}/\mathsf{tor}(\pi^*\mathcal{J})$ to the component L is $\mathcal{I}^k_{L,\tilde{s}} \subset \mathcal{O}_{L,\tilde{s}}$, the $k-th$ power of the ideal sheaf of some point $\tilde{s} \in L$. Let $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu}) \in \mathsf{Tr}_X$ be a simple triple. Then $\widetilde{\mathcal{F}}|_L = \oplus_{j=1}^k \mathcal{O}_L(n+j)^{r_j}$ for some $n \in \mathbb{Z}$.*

Proof. Assume $\pi^*\mathcal{F}|_L$ contains a summand $\mathcal{O}_L(n) \oplus \mathcal{O}_L(m)$ with $m > n+k$. Then there is a nonzero homogeneous form

$$Q \in \mathrm{Hom}_{\Bbbk}(\mathcal{O}_L(n), \mathcal{O}_L(m)) \cong \Bbbk[z_0, z_1]_{m-n}$$

such that $\tilde{\imath}^* Q = 0$. The rest follows from Lemma 2.6.2.

Indeed, a nontrivial endomorphism (F, f) of a triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ can be constructed as follows: take $f = 0$, choose a matrix $F|_L$ in the matrix form (2.6) such that there exists a nonzero homogeneous form Q on some entry of the block (m, n) and all other entries are zeros, and $F|_{L'} = 0$ for all other components $L' \neq L$. Although the endomorphism $(\tilde{\imath}^* F, \tilde{\pi}^* f)$ of $\tilde{\mu}$ is zero, the pair (F, f) induces a nontrivial endomorphism of $\mathcal{F}$. Thus we obtain a contradiction to the assumption that $\mathcal{F}$ is simple. □

In particular, for curves of arithmetic genus one such as: Kodaira fibers of types II,III, IV and curves consisting of $N+1$ generic concurrent lines in $\mathbb{P}^N$, Lemma 2.6.4 implies that for a simple torsion free sheaf $\mathcal{F}$ and each component $L := L_k$

$$\big(\pi^*\mathcal{F}/\mathsf{tor}(\pi^*\mathcal{F})\big)|_L = \big(\mathcal{O}_L(n_k)\big)^{r_{n_k}} \oplus \big(\mathcal{O}_L(n_k+1)\big)^{r_{n_k+1}}. \tag{2.19}$$

Thus later on for simple torsion free sheaves on degenerated elliptic curves we assume that the matrix $\tilde{\mu}$ restricted to the component L contains at most two blocks 0 and 1, and moreover, the map (2.8) is an isomorphism.

2.7 Category of block matrices BM_P

Let E be a plane reduced cubic curve with N irreducible components ($1 \le N \le 3$). By Theorem 2.1.3 a torsion free sheaf on E is described by a triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$. In Section 2.3 we replaced the category of triples Tr_E by the category of matrices MP_E. Objects of MP_E are maps $\tilde{\mu}$ given by $2N$ matrices over $\Bbbk$. This matrix problem is quite cumbersome, moreover for Kodaira fibers II, III and IV it is wild. But if we are interested in a description of simple torsion free sheaves, it follows that the matrix $\tilde{\mu}$ restricted to each irreducible component contains at most two horizontal blocks. Then it turns out that $2N-1$ matrices can be reduced to a canonical form consisting of identity and zero blocks. The set of transformations for the last matrix, keeping the remaining $2N-1$ matrices unchanged, leads to a quite special class of matrix problems, denoted by BM_P (standing for *block-matrices* of the form determined by a poset P). We shall carry out the detailed calculations for each Kodaira fiber in the corresponding sections. Now let us explain the final result.

Let $I := I' \cup I'' := \{1, \dots, n\}$ be a set of indices, and " $\prec$ " be a partial order on I. A *poset* $P = (I, \prec)$ is a set of pairs

$$P = \{(i,j) | \ \ i \prec j\} \subset I \times I.$$

It is convenient to visualize P as an oriented graph with two types of vertices. Elements of I' will be denoted by bullets and elements of I'' by circles. The set of arrows is defined by the partial order: $i \to j$ for $i \succ j$.

Let us finally formulate the matrix problem for a given poset P. For a Kodaira fiber of type II, III or IV we define a category BM_P :

$$\mathsf{BM}_P = \bigcup_{\mathbb{s}} \mathsf{BM}_P(\mathbb{s}), \tag{2.20}$$

where $\mathbb{s} := (s_1, \dots, s_n) \in (\mathbb{N} \cup \{0\})^n$ is a tuple of sizes. Objects of $\mathsf{BM}_P(\mathbb{s})$ are block-matrices

$$B = (B_{ij}), \text{ where } (i,j) \in P \cup \operatorname{diag}(I' \times I') \text{ and } B_{ij} \in \mathsf{Mat}_{s_i \times s_j}$$

and morphisms $S : B \to B'$ are block matrices

$$S = (S_{ij}), \text{ where } (i,j) \in P^t \cup \operatorname{diag}(I \times I) \text{ and } S_{ij} \in \mathsf{Mat}_{s_i \times s_j}$$

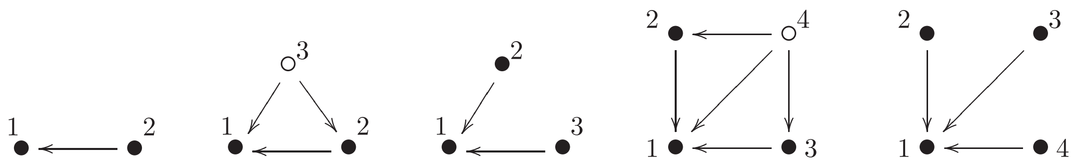
for $P^t := \{(i,j)|\ j \prec i\} = \{(i,j)|\ (j,i) \in P\}$ being the dual poset to P; and satisfying equations:

$$SB|_{P \cup \operatorname{diag}(I' \times I')} = B'S|_{P \cup \operatorname{diag}(I' \times I')}.$$

Two matrices B and $B' \in \mathsf{BM}_P(\mathbb{s})$ are *equivalent* if there exists an invertible morphism $S : B \to B'$. A matrix $B \in \mathsf{BM}_P$ is called *simple* or a *brick,* if any endomorphism $S : B \to B$ is scalar. The full subcategory of simple objects is denoted by BM_P^s.

Remark 2.7.1. Let P be a poset which describes a matrix problem BM_P for vector bundles, then $I = I'$, and all vertices are bullets.

Example 2.7.2. Let us list some posets, which characterize matrix problems (i)-(v) up to some modifications. However we should mention that in coarse of matrix reduction we will meet more matrix problems determined by some posets P.



(i) (ii) (iii) (iv) (v)

Example 2.7.3. Let us present matrices $B \in \mathsf{BM}_P$ for posets from Example 2.7.2.

$$\begin{array}{c|c|c|}
 & 1 & 2 \\ \hline
1 & * & * \\ \hline
2 & & * \\ \hline
\end{array}
\qquad
\begin{array}{c|c|c|c|}
 & 1 & 2 & 3 \\ \hline
1 & * & * & * \\ \hline
2 & & * & * \\ \hline
3 & & & \\ \hline
\end{array}
\qquad
\begin{array}{c|c|c|c|}
 & 1 & 2 & 3 \\ \hline
1 & * & * & * \\ \hline
2 & & * & \\ \hline
3 & & & * \\ \hline
\end{array}
\qquad
\begin{array}{c|c|c|c|c|}
 & 1 & 2 & 3 & 4 \\ \hline
1 & * & * & * & * \\ \hline
2 & & * & & * \\ \hline
3 & & & * & * \\ \hline
4 & & & & \\ \hline
\end{array}
\qquad
\begin{array}{c|c|c|c|c|}
 & 1 & 2 & 3 & 4 \\ \hline
1 & * & * & * & * \\ \hline
2 & & * & & \\ \hline
3 & & & * & \\ \hline
4 & & & & * \\ \hline
\end{array}$$

(i) (ii) (iii) (iv) (v)

where stars "$*$" and empty space denote respectively B_{ij} and zero blocks.

Recall that the triple approach induces a full and dense functor: $\mathsf{TF}_X \xrightarrow{\sim} \mathsf{Tr}_X \longrightarrow \mathsf{MP}_X$ and the primary reduction induces $\mathsf{MP}_X^s \xrightarrow{\sim} \mathsf{BM}_P^s$. It turns out that for simple vector bundles the composition of this functors imposes an equivalence

$$\mathsf{VB}_E^s(r, \mathbb{d}) \xrightarrow{\sim} \mathsf{BM}_P^s(\mathbb{s}), \tag{2.21}$$

for some special poset P and a tuple of sizes $\mathbb{s}$, where the category BM_P and the tuple $\mathbb{s}$ are uniquely defined by the curve E, the rank r and the multidegree $\mathbb{d}$.

Remark 2.7.4. There is also a one-to-one correspondence $\mathsf{TF}_E^s \xrightarrow{\sim} \mathsf{BM}_P^s$, but the functors should be constructed separately for different tuples $(r, \mathbb{d})$ and depending on whether $\mathcal{F} \in \mathsf{TF}_E^s$ is a vector bundle or a torsion free sheaf.

Matrix reduction in terms of categories BM_P

It would be nice to have a step of matrix reduction as an equivalence: $\mathsf{BM}_P^s(\mathbb{s}) \xrightarrow{\sim} \mathsf{BM}_{P'}^s(\mathbb{s}')$ where $P, P' \subset I \times I$ and $\mathbb{s}' < \mathbb{s}$ (i.e. $s_i' \leq s_i$ for all $i \in I$ and there exists at least one i such that $s_i' < s_i$). It is indeed the case for posets on three or fewer vertices. Unfortunately, it can happen that morphisms of the new category can not be expressed using the matrix multiplication anymore. Therefore, we are forced to replace $\mathsf{BM}_{P'}$ by the category of representations $\mathsf{Rep}(Q, \partial)$ for some differential biquiver $(Q, \partial)_{P'}$ determined by a poset P'. We treat the category $\mathsf{Rep}(Q, \partial)$ in a formal way in Chapter 7.

2.8 Simple vector bundles on a nodal cubic curve

The problem of describing simple vector bundles on a nodal cubic curve can be considered separately from the description of all indecomposable objects. As we shall see below for a nodal cubic curve E this problem is *self-reproducing* and can be solved without using the combinatorics of strings and bands [BD04].

First of all note that for a simple triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ Lemma 2.6.3 implies

$$\widetilde{\mathcal{F}} \cong \widetilde{\mathcal{O}}(c)^{r_1} \oplus \widetilde{\mathcal{O}}(c+1)^{r_2} \tag{2.22}$$

for some $c \in \mathbb{Z}$ and the matrix $\tilde{\mu} = (\mu(0), \mu(\infty))$ consists of two horizontal blocks. As was mentioned above both matrices $\mu(0)$ and $\mu(\infty)$ are invertible. One of them, let us say, $\mu(0)$ can be reduced to the identity form

$$\mu(0) = \mathbb{I}_r.$$

From the equation (2.17), we obtain $f = F(0)$ and the following reduced matrix problem:

Reduced matrix problem for simple vector bundles

Thus we obtain a new matrix problem for the matrix $\mu(\infty)$ given by the following transformations:

$$\mu(\infty) \mapsto F(\infty)\mu(\infty)F(0)^{-1}.$$

The permitted transformations are listed below:

1. An arbitrary invertible elementary transformation simultaneously for the first block-row and the first block-column of $\mu(\infty)$.
2. An arbitrary invertible elementary transformation simultaneously in the second block-row and the second block-column of $\mu(\infty)$.
3. We can add the first block-row of $\mu(\infty)$ to the second one.
4. We can add the second block-column of $\mu(\infty)$ to the first one.

This matrix problem corresponds to the category of square matrices divided into blocks

$$\mathsf{BM}_{\widetilde{P}} = \bigcup_{(r_1,r_2)} \mathsf{BM}_{\widetilde{P}}(r_1, r_2), \tag{2.23}$$

where $(r_1, r_2) \in \mathbb{Z}^2_{\geq 0}$. Objects of $\mathsf{BM}_{\widetilde{P}}(r_1, r_2)$ are square matrices

$$B := \mu(\infty) = \begin{array}{|c|c|} \hline B_1 & B_{12} \\ \hline B_{21} & B_2 \\ \hline \end{array}$$

consisting of the blocks $(B_1, B_{12}, B_{21}, B_2)$, where (B_1, B_2) are square matrices of sizes r_1 and r_2 respectively. A morphism $S : B \to B'$ is given by two lower triangular block matrices:

$$S = \left(\begin{array}{|c|c|} \hline S_1 & 0 \\ \hline S_{21} & S_2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline S_1 & 0 \\ \hline S'_{21} & S_2 \\ \hline \end{array} \right)$$

with block sizes (r_1, r_2) and satisfying equations for blocks

$$\begin{array}{rcl} S_1B_1 & = & B'_1S_1 + B'_{12}S'_{21}, \\ S_1B_{12} & = & B'_{12}S_2, \\ S_{21}B_{12} + S_2B_2 & = & B'_2S_2, \\ S_{21}B_1 + S_2B_{21} & = & B'_{21}S_1 + B'_2S'_{21}. \end{array} \tag{2.24}$$

Two matrices B and B' are called *equivalent* (i.e. correspond to isomorphic vector bundles) if there is a non-degenerate morphism $S : B \to B'$, i.e. if $B' = SBS^{-1}$. A matrix $B \in \mathsf{BM}_{\widetilde{P}}(r_1, r_2)$ is called *simple* if any endomorphism $S : B \to B$ is scalar. Obviously, the simplicity is a property defined on equivalence classes. As usual $\mathsf{BM}^{s}_{\widetilde{P}}(r_1, r_2)$ denotes the subcategory of $\mathsf{BM}_{\widetilde{P}}(r_1, r_2)$ consisting of simple objects.

We start with reduction of the block B_{12}. Assume B_{12} has a zero-row k and a zero-column j. Then by a transformation S'_{21} add the column j to the column k and by the proper transformation S_{21} add the multiple of the row k to the row j, so that the block B_{21} remains unchanged. In such a way we construct a nonscalar endomorphism. More detailed: assume B is reduced to the form:

$$B = \begin{array}{|c|c|c|c|}\hline X_1 & X_2 & 0 & 0 \\ \hline 0 & 0 & \mathbb{I} & 0 \\ \hline Z_1 & Z_{12} & 0 & Y_1 \\ \hline Z_{21} & Z_2 & 0 & Y_2 \\ \hline \end{array}$$

for some nonreduced blocks X_i, Y_j and Z_{ij}, where $i, j \in \{1, 2\}$. Then a nonscalar endomorphism has the form:

$$S = \begin{array}{|c|c|c|c|}\hline \mathbb{I} & 0 & 0 & 0 \\ \hline 0 & \mathbb{I} & 0 & 0 \\ \hline Y_1 & 0 & \mathbb{I} & 0 \\ \hline Y_2 & 0 & 0 & \mathbb{I} \\ \hline \end{array} \quad \text{and } S' = \begin{array}{|c|c|c|c|}\hline \mathbb{I} & 0 & 0 & 0 \\ \hline 0 & \mathbb{I} & 0 & 0 \\ \hline 0 & 0 & \mathbb{I} & 0 \\ \hline X_1 & X_2 & 0 & \mathbb{I} \\ \hline \end{array}.$$

Thus for a simple object B one can assume that the block B_{12} has maximal rank. Assume $r_1 = r_2$ and B_{12} is reduced to the identity form. Having $B_{12} = \mathbb{I}$ we can "kill" both blocks B_1 and B_2, to the zero form and reduce the block B_{21} to a Jordan normal form $J(\lambda)$, for $\lambda \in \Bbbk$. Since $B = \mu(\infty)$ is invertible by definition, it implies $\lambda \neq 0$. If $r_2 = 1$, then $B_{21} = \boxed{\lambda}$, for $\lambda \in \Bbbk^*$. It is easy to check that such B is simple, However, for $r_2 > 1$ the Jordan normal form has an endomorphism, which can be extended to an endomorphism of B.

Therefore, if B is simple, then B_{12} can be reduced to one of the following forms:

$$B_{12} = \begin{cases} \boxed{\begin{array}{c} 0 \\ \mathbb{I}_{r_2} \end{array}} & \text{if} \quad r_1 > r_2, \\ \boxed{\mathbb{I}_{r_1} 0} & \text{if} \quad r_2 > r_1, \\ \boxed{1} & \text{if} \ \ r_1 = r_2 = 1. \end{cases} \tag{2.25}$$

Since we have all ingredients for small matrix reduction, let us present an algorithm for constructing a canonical form.

Algorithm 2.8.1. Let $(r, d) \in \mathbb{N} \times \mathbb{Z}$ be a pair of coprime integers, and $\lambda \in \Bbbk$.

- First, by the Euclidean algorithm we find integers c, r_1 and r_2, $0 < r_1 \le r$, $0 \le r_2 < r$ such that $cr + r_2 = d$ and $r_1 + r_2 = r$. Thus we recover the normalization sheaf $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}}(c)^{r_1} \oplus \widetilde{\mathcal{O}}(c+1)^{r_2}$, and sizes of blocks (r_1, r_2).
- If $r = r_1 = 1$ then $\mathcal{F}$ is a line bundle and $B = \lambda \in \Bbbk^*$
- If $r_1 = r_2 = 1$ then

$$B(\lambda) = \begin{array}{|c|c|l} \hline 0 & 1 & c \\ \hline \lambda & 0 & c{+}1 \\ \hline \end{array} \qquad \lambda \in \Bbbk^*.$$

Using this input data we construct the matrix $B(\lambda) \in \mathsf{BM}^s_P(r_1, r_2)$ inductively:

- Assume we have the matrix $B_1(\lambda) \in \mathsf{BM}^s_P(r_1, r_2)$,

$$B_1(\lambda) = \begin{array}{|c|c|} \hline X & Y \\ \hline W & Z \\ \hline \end{array}$$

then $B(\lambda) \in \mathsf{BM}^s_{\widetilde{P}}(r_1 + r_2, r_2)$ has form

$$B(\lambda) = \begin{array}{|c|c|c|} \hline X & Y & 0 \\ \hline 0 & 0 & \mathbb{I}_{r_2} \\ \hline W & Z & 0 \\ \hline \end{array}$$

and respectively, $B(\lambda) \in \mathsf{BM}_{\widetilde{P}}(r_1, r_1 + r_2)$:

$$B(\lambda) = \begin{array}{|c|c|c|} \hline 0 & \mathbb{I}_{r_1} & 0 \\ \hline X & 0 & Y \\ \hline W & 0 & Z \\ \hline \end{array}.$$

- Finally, we get the matrix $\tilde{\mu} = (\mu(0), \mu(\infty)) = (\mathbb{I}_r, B(\lambda))$.

We postpone the proof of the validness of the matrix reduction till Chapter 6, where it will be carried out under general assumptions using the formalism of bocses. Summarizing, we obtain the following classification.

Theorem 2.8.2. *Let E be a nodal cubic curve over an algebraically closed field $\Bbbk$. Then*

1. *the rank r and the degree d of a simple torsion free sheaf $\mathcal{F}$ over E are coprime;*

2. *a simple vector bundle is determined by its rank, degree and a continuous parameter $\lambda \in \Bbbk^* \cong E_{reg}$.*

This description can be alternatively given in terms of bunches of chains i.e. recovered from the description of all indecomposable vector bundles [Bur03, BBDG].

Chapter 3

Vector bundles and torsion free sheaves on a cuspidal cubic curve

This chapter is devoted to a classification of torsion free sheaves on a cuspidal cubic curve. As was mentioned in the introduction, the problem of describing all indecomposable torsion free sheaves is representation-wild, but if we restrict ourselves to the subcategory of simple (stable) torsion free sheaves TF^s_E, then the classification problem becomes tame again. The combinatorics of the answer resembles the case of smooth and nodal Weierstraß curves (see Theorem 1.0.1 and Theorem 2.8.2). The main result of this chapter is the following.

Theorem 3.0.1. *Let E be a cuspidal cubic curve over an algebraically closed field $\Bbbk$. Then*

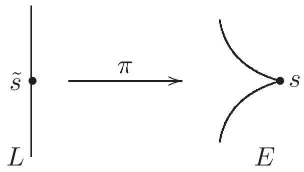
1. *the rank r and the degree d of a simple torsion free sheaf $\mathcal{F}$ over E are coprime;*

2. *for every coprime pair $(r, d) \in \mathbb{N} \times \mathbb{Z}$, the fine moduli space of simple torsion free sheaves $\mathsf{TF}^s_E(r, d)$ is isomorphic to the curve E itself. Moreover, vector bundles correspond to regular points $E_{reg} \cong \mathsf{Pic}^0_E \cong \mathbb{A}^1$ and there is a unique torsion free not locally free sheaf corresponding to the singular point of E.*

This result can also be obtained by other methods for example by using Fourier-Mukai transforms or by the GPB approach. However, here we not only describe fine moduli spaces of simple torsion free sheaves but also give an explicit description of a universal family of simple vector bundles with prescribed rank and degree. To be precise: Algorithm 3.2.2 constructs a canonical form of the “gluing” matrix of the triple corresponding to a vector bundle $\mathcal{E}$ of given rank r, degree d (g.c.d.(r,d)=1) and determinant $\det(\mathcal{E}) \in \mathsf{Pic}^d_E$; Algorithm 3.3.1 constructs a canonical form of the matrix corresponding to a unique torsion free and not locally free sheaf of given rank and degree (which should be coprime again). In Section 3.6 we propose a construction of a universal bundle of $\mathsf{VB}^s_E(r, d)$.

3.1 Reduction to a matrix problem

Let E be a cuspidal cubic curve in $\mathbb{P}^2$ given by the equation $x^3 - y^2 z = 0$. Choose coordinates $(z_0 : z_1)$ on the normalization $\widetilde{E} \cong \mathbb{P}^1 \xrightarrow{\pi} E$ such that the preimage of the singular point $s = (0:0:1)$ of E is $0 := (0:1)$.



Let $U = \{(z_0 : z_1) | z_1 \neq 0\}$ be an affine neighborhood of 0 and $z = z_0/z_1$. In the notations of Section 2 we have: $\mathcal{O}_S \cong \Bbbk(s)$ and $\mathcal{O}_{\widetilde{S}} \cong \big(\Bbbk[\varepsilon]/\varepsilon^2\big)(s)$.
Analogously to the case of a nodal rational curve, for a triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ we fix:

- a splitting $\widetilde{\mathcal{F}} \cong \bigoplus\limits_{n \in \mathbb{Z}} \widetilde{\mathcal{O}}(n)^{r_n}$, with $\sum_{n\in\mathbb{Z}} r_n = r$;
- an isomorphism $\mathcal{M} \cong \mathcal{O}_S^{r+t} \cong \Bbbk^{r+t}$, for some $t \geq 0$, where $t = 0$ if and only if $\mathcal{F}$ is a vector bundle;
- a set of trivializations $\widetilde{\mathcal{O}}(n) \otimes \mathcal{O}_{\widetilde{S}} \longrightarrow \big(\Bbbk[\varepsilon]/\varepsilon^2\big)(s)$ given by the map $\zeta \otimes 1 \mapsto pr(\frac{\zeta}{z_1^n})$ for a local section ζ of $\widetilde{\mathcal{O}}(n)$ on U, where $pr : \Bbbk[U] \longrightarrow \Bbbk[\varepsilon]/\varepsilon^2$ is the map induced by $\Bbbk[z] \longrightarrow \Bbbk[\varepsilon]/\varepsilon^2$, $z \mapsto \varepsilon$.

With respect to all these choices the morphisms $\tilde{\mu}$, $\tilde{\imath}^* F$ and $\tilde{\pi}^* f$ can be written as matrices.

- An epimorphism of $\Bbbk[\varepsilon]/\varepsilon^2$–modules $\tilde{\mu} : \tilde{\pi}^*\mathcal{M} \longrightarrow \tilde{\imath}^*\widetilde{\mathcal{F}}$, (which is an isomorphism if and only if $\mathcal{F}$ is a vector bundle) can be written as
$$\tilde{\mu} = \mu(0) + \varepsilon\mu_\varepsilon(0), \tag{3.1}$$
where both $\mu(0)$ and $\mu_\varepsilon(0)$ are $r \times (r+t)$ matrices (square in case of vector bundles). Since by Theorem 2.1.3 the isomorphism classes of triples stand in bijection with the isomorphism classes of torsion free sheaves, we have to study the action of automorphisms of $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ on the matrices $\mu(0)$ and $\mu_\varepsilon(0)$.
- If a morphism $\widetilde{\mathcal{O}}(n) \to \widetilde{\mathcal{O}}(m)$ is given by a homogeneous form $Q(z_0, z_1)$ of degree $m-n$, then the induced map $\widetilde{\mathcal{O}}(n) \otimes \mathcal{O}_{\widetilde{S}} \longrightarrow \widetilde{\mathcal{O}}(m) \otimes \mathcal{O}_{\widetilde{S}}$ is given by the map
$$pr(Q(z_0, z_1)/z_1^{m-n}) = Q(0:1) + \varepsilon\tfrac{dQ}{dz_0}(0:1).$$

Hence, for any endomorphism (F, f) of the triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ the induced map
$\tilde{\imath}^* F : \tilde{\imath}^* \widetilde{\mathcal{F}} \longrightarrow \tilde{\imath}^* \widetilde{\mathcal{F}}$ has the form

$$\tilde{\imath}^* F = F(0) + \varepsilon \tfrac{dF}{dz_0}(0), \tag{3.2}$$

where, as usual, we write $F(0)$ for $F(0:1)$. If (F, f) is an automorphism, then $\tilde{\imath}^* F = F(0) + \varepsilon \frac{dF}{dz_0}(0) \in \mathrm{GL}(\Bbbk[\varepsilon]/\varepsilon^2, r)$;

- $\tilde{\pi}^* f = f \in \mathrm{Mat}_{\Bbbk}(r \times r)$ and if (F, f) is an isomorphism then $f \in \mathrm{GL}(\Bbbk, r)$.

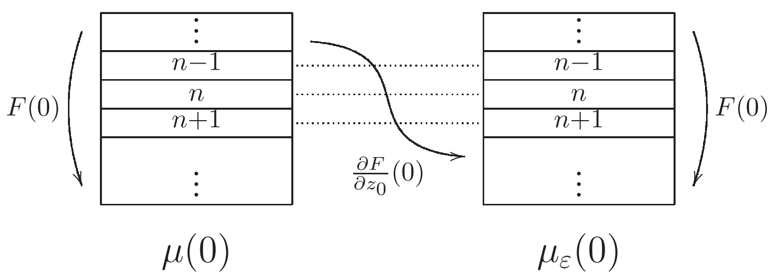
We obtain the following matrix transformations:

$$\begin{cases} \mu(0) & \mapsto F(0)\mu(0)f^{-1}, \\ \mu_\varepsilon(0) & \mapsto F(0)\mu_\varepsilon(0)f^{-1} + \frac{dF}{dz_0}(0)\mu(0)f^{-1} \end{cases} \tag{3.3}$$

The condition for $\tilde{\mu}$ to be surjective is equivalent to the surjectivity of $\mu(0)$. Similarly, $\tilde{\mu}$ is invertible if and only if $\mu(0)$ is invertible.

Original matrix problem for a cuspidal cubic curve

As a result, the matrix problem MP_E describing torsion free sheaves on a cuspidal cubic curve is as follows: we have two matrices $\mu(0)$ and $\mu_\varepsilon(0)$ with r rows and $r+t$ columns, and $rank(\mu(0)) = r$. In the subproblem corresponding to vector bundles the matrices $\mu(0)$ and $\mu_\varepsilon(0)$ are square and $\mu(0)$ is invertible. Moreover, $\mu(0)$ and $\mu_\varepsilon(0)$ are divided into horizontal blocks labelled by integers, also called weights. Any two blocks of $\mu(0)$ and $\mu_\varepsilon(0)$ marked by the same label are called *conjugated* and have the same number of rows.



We are allowed to perform the following transformations.

1. An arbitrary elementary invertible transformation of columns simultaneously for the matrices $\mu(0)$ and $\mu_\varepsilon(0)$. Such transformations correspond to the matrix f.

2. An arbitrary invertible elementary transformations of rows of $\mu(0)$ and $\mu_\varepsilon(0)$ simultaneously inside of any two conjugated horizontal blocks. Such transformations correspond to the diagonal blocks of the matrix $F(0)$.

3. We can add a scalar multiple of any row with a lower weight to any row with a higher weight simultaneously in $\mu(0)$ and $\mu_\varepsilon(0)$. Such transformations correspond to the non-diagonal blocks of the matrix $F(0)$.

4. We can add a row of $\mu(0)$ with a lower weight to a row of $\mu_\varepsilon(0)$ with a higher weight. Such transformations correspond to (non-diagonal) blocks of the matrix $\frac{dF}{dz_0}(0)$.

This matrix problem turns out to be wild even for two horizontal blocks, see [Dro92, Section 1] and [DG01, Section 6]. However, the simplicity condition of a triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ imposes some additional restrictions, which make the problem tame. In particular for a simple torsion free sheaf $\mathcal{F}$ Lemma 2.6.4 implies

$$\widetilde{\mathcal{F}} \cong \widetilde{\mathcal{O}}(c)^{r_1} \oplus \widetilde{\mathcal{O}}(c+1)^{r_2} \tag{3.4}$$

for some $c \in \mathbb{Z}$, and thus the matrix $\tilde{\mu}$ consists of only two horizontal blocks.

The matrix problem for simple objects is similar to the analogous problem for a nodal curve considered in Section 2.8. It can be solved for simple objects without use of any additional technique. In Section 6 we treat the same problem formally, and in Example 6.5.6 we present the matrix reduction in the language of bocses.

3.2 Matrix problem for simple vector bundles

We consider the case of vector bundles first. Although the general case of torsion free not locally free sheaves is similar, we decided to consider it separately in order to make the presentation clearer.

As already mentioned, if $\mathcal{F}$ is a vector bundle, then $\tilde{\mu}$ is an isomorphism and the matrix $\mu(0)$ is invertible too, and by transformations 1 and 2 the matrix $\mu(0)$ can be reduced to the identity matrix. Moreover, using transformations 4 we can make the left lower block of $\mu_\varepsilon(0)$ zero, as indicated below:

$$\mu(0) = \begin{array}{|c|c|}\hline \mathbb{I}_{r_1} & 0 \\ \hline 0 & \mathbb{I}_{r_2} \\ \hline \end{array} \quad \text{and} \quad \mu_\varepsilon(0) = \begin{array}{|c|c|}\hline B_1 & B_{12} \\ \hline 0 & B_2 \\ \hline \end{array}. \tag{3.5}$$

Here $\mathbb{I}_n$ denotes the identity matrix of size n and B_1, B_{12}, B_2 are three nonreduced blocks. To preserve the identity form of the matrix $\mu(0)$ for the later investigations we should assume $F(0) = f$, and hence, f inherits the same lower-block-triangular structure as $F(0)$.

Reduced matrix problem for simple vector bundles

We obtain a new matrix problem for the matrix $\mu_\varepsilon(0)$ which reads:

$$\mu_\varepsilon(0) \mapsto F(0)\mu_\varepsilon(0)F(0)^{-1}.$$

The allowed transformations are listed below:

1. An arbitrary invertible elementary transformation simultaneously in the first block-row and the first block-column of $\mu_\varepsilon(0)$.
2. An arbitrary invertible elementary transformation simultaneously in the second block-row and the second block-column of $\mu_\varepsilon(0)$.
3. We can add a row of the first block-row of $\mu_\varepsilon(0)$ to a row of the second one and simultaneously subtract the corresponding column of the second block-column from the corresponding column of the first one.

Block matrix category

This matrix problem can be presented as a category of block matrices BM_P, for the set of indices $I = \{1,2\}$ and $1 \prec 2$ a poset $P = (I, \prec) = \{(1,2)\} \subset \{1,2\} \times \{1,2\}$ in agreement with notations from Section 2.7. Then

$$\mathsf{BM}_P = \bigcup_{(r_1,r_2)} \mathsf{BM}_P(r_1,r_2), \tag{3.6}$$

where $(r_1,r_2) \in \mathbb{N} \times \mathbb{N}$. Objects of $\mathsf{BM}_P(r_1,r_2)$ are matrices of the form $\mu_\varepsilon(0)$ in formula (3.5), i.e. upper-block-triangular matrices B consisting of the blocks (B_1, B_{12}, B_2), where (B_1, B_2) are square matrices of sizes r_1 and r_2 respectively. Morphisms $S : B \to B'$ are given by lower-block-triangular matrices:

$$S = \begin{array}{|c|c|} \hline S_1 & 0 \\ \hline S_{21} & S_2 \\ \hline \end{array}$$

with sizes of blocks (r_1,r_2) (i.e. $B_i, S_i \in \mathrm{Mat}_{\Bbbk}(r_i \times r_i)$, for $i = 1,2$) and satisfying the equation $SB = B'S$ modulo the left lower block B_{21}. It can be rewritten as a system of equations for blocks:

$$\begin{array}{rcl} S_1B_1 &=& B'_1S_1 + B'_{12}S_{21}, \\ S_1B_{12} &=& B'_{12}S_2, \\ S_2B_2 + S_{21}B_{12} &=& B'_2S_2. \end{array} \tag{3.7}$$

Remark 3.2.1. In terms of Section 6 the category BM_P is the category of representations of the differential biquiver (Q, ∂) from Example 6.5.6.

Assume the block B_{12} has a zero-row k and a zero-column j. Then taking proper S_{21} we can add column j to column k and simultaneously subtract row k from row j. The matrix B remains unchanged under this transformation. More detailed: assume B is reduced to the form

$$B = \begin{array}{|c|c|c|c|}\hline X_1 & X_2 & 0 & 0 \\ \hline 0 & X_3 & \mathbb{I} & 0 \\ \hline & & 0 & Y_1 \\ \cline{3-4} & & 0 & Y_2 \\ \hline \end{array}\,.$$

Then a nonscalar endomorphism has form:

$$S = \begin{array}{|c|c|c|c|}\hline \mathbb{I} & 0 & 0 & 0 \\ \hline 0 & \mathbb{I} & 0 & 0 \\ \hline 0 & 0 & \mathbb{I} & 0 \\ \hline T & 0 & 0 & \mathbb{I} \\ \hline \end{array}$$

for an arbitrary nonzero block T. Hence, we obtain a nonscalar endomorphism. Thus for a simple object B we can assume that the block B_{12} has the maximal rank. Then for $r_1 = r_2$ the block B_{12} is a square matrix and can be reduced to the identity matrix $\mathbb{I}$. Having $B_{12} = \mathbb{I}$ we can make one of matrices B_1 and B_2, say B_1, zero and the other one B_2 can be reduced to its Jordan normal form. If $r_2 = 1$, then $B_2 = \boxed{\lambda}$, $\lambda \in \Bbbk$, in this case B is simple, but for $r_2 > 1$ the Jordan normal form has an endomorphism, which can be extended to an endomorphism of B. Therefore, if B is simple, then B_{12} can be reduced to one of the following forms

$$B_{12} = \begin{cases} \boxed{\begin{matrix} 0 \\ \mathbb{I}_{r_2} \end{matrix}} & \text{if} \quad r_1 > r_2, \\ \boxed{\mathbb{I}_{r_1} 0} & \text{if} \quad r_2 > r_1, \\ \boxed{1} & \text{if} \ r_1 = r_2 = 1, \end{cases} \tag{3.8}$$

From the system of equations (3.7) we get that in the case $r_1 > r_2$ block the B_2 can be reduced to the zero matrix and the block B_1 to the upper triangular block-matrix formed by three nonzero subblocks $(B_{1.1}, B_{1.12}, B_{1.2})$. Long but straightforward calculations show that the transformations of B which preserve already reduced blocks are uniquely determined by the automorphisms of B_1 in the category BM_P^s. Moreover, $\mathrm{End}_{\mathsf{BM}_P}(B_1) = \mathrm{End}_{\mathsf{BM}_P}(B)$. In Section 6 we give rigorous statements about such kinds of reduction, and in Example 6.5.6 this matrix problem will be treated formally.

In the same way the matrix B can be reduced in the case $r_2 > r_1$. Thus the problem $\mathsf{BM}_P^s(r_1, r_2)$ is *self-reproducing,* that means that the poset P, and hence the problem BM_P defined by equations (3.6) remain unchanged under the matrix reduction. To be precise, we get a bijection between $\mathsf{BM}_P^s(r_1, r_2)$ and $\mathsf{BM}_P^s(r_1 - r_2, r_2)$ if $r_1 > r_2$, between $\mathsf{BM}_P^s(r_1, r_2)$ and $\mathsf{BM}_P^s(r_1, r_2 - r_1)$ if $r_2 > r_1$. Finally, if $r_1 = r_2 > 1$ then $\mathsf{BM}_P^s(r_1, r_1)$ is empty.

In this reduction procedure one can easily recognize the Euclidean algorithm. Moreover, the reduction terminates after finitely many steps when we achieve $r_1 = r_2 = 1$. Without loss of generality we may assume that the matrix $B \in \mathsf{BM}_P^s(1, 1)$ has the form

$$B = \begin{array}{|c|c|}\hline 0 & 1 \\ \hline & \lambda \\ \hline \end{array}. \tag{3.9}$$

Note that this matrix form is equivalent to the matrices $\begin{array}{|c|c|}\hline \lambda & 1 \\ \hline & 0 \\ \hline \end{array}$ and $\begin{array}{|c|c|}\hline \frac{1}{2}\lambda & 1 \\ \hline & \frac{1}{2}\lambda \\ \hline \end{array}$. Objects of $\mathsf{BM}_P^s(1, 1)$ are parametrized by a continuous parameter $\lambda \in \Bbbk$, thus the same holds for $\mathsf{BM}_P^s(r_1, r_2)$ with coprime r_1 and r_2.

We interpret sizes of blocks r_1 and r_2 in terms of geometric invariants of a vector bundle. From formulas (3.4) and (2.12) it follows immediately that $r_1 + r_2 = r$ and $r_2 = d\,\mathsf{BM}_P\, odr$. Hence, the matrix reduction can be written in terms of rank and degree. Let us present it as an algorithm.

Algorithm 3.2.2. Let $(r, d) \in \mathbb{N} \times \mathbb{Z}$ be a pair of coprime integers, and $\lambda \in \Bbbk$.

- First, by the Euclidean algorithm we find integers c, r_1 and r_2, $0 < r_1 \leq r$, $0 \leq r_2 < r$ such that $cr + r_2 = d$ and $r_1 + r_2 = r$. Thus we recover the normalization sheaf $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}}(c)^{r_1} \oplus \widetilde{\mathcal{O}}(c+1)^{r_2}$, and sizes of blocks (r_1, r_2).
- If $r_1 = r_2 = 1$ the matrix $B(\lambda)$ has form (3.9).

Using this input data we construct the matrix $B(\lambda) \in \mathsf{BM}_P^s(r_1, r_2)$ inductively reversing Euclidean reduction:

- Assume there is a matrix $B_1(\lambda) \in \mathsf{BM}_P^s(r_1, r_2)$ of the form

$$B_1(\lambda) = \begin{array}{|c|c|}\hline X & Y \\ \hline & Z \\ \hline \end{array}$$

  then $B(\lambda) \in \mathsf{BM}_P^s(r_1 + r_2, r_2)$ has form

$$B(\lambda) = \begin{array}{|c|c|c|}\hline X & Y & 0 \\ \hline 0 & Z & \mathbb{I}_{r_2} \\ \hline \multicolumn{2}{|c|}{} & 0 \\ \hline \end{array}.$$

and respectively, $B(\lambda) \in \mathsf{BM}_P(r_1, r_1 + r_2)$ is

$$B(\lambda) = \begin{array}{|c|c|c|}\hline 0 & \mathbb{I}_{r_1} & 0 \\ \hline & X & Y \\ \cline{2-3} & 0 & Z \\ \hline \end{array}.$$

- Finally, we get the matrix $\tilde{\mu} = \mu(0) + \varepsilon\mu_\varepsilon(0) = \mathbb{I}_r + \varepsilon B(\lambda)$.

Let us illustrate this on a small example:

Example 3.2.3. Let $\mathcal{E} \in \mathsf{VB}^s_E(7, 12)$ be a simple vector bundle of rank 7 and degree 12. Let us find a canonical form of the matrix $\mu_\varepsilon(0)$. First, we calculate the normalization sheaf $\widetilde{\mathcal{E}} = \widetilde{\mathcal{O}}(1)^2 \oplus \widetilde{\mathcal{O}}(2)^5$. Thus, in our notations $r_1 = 2$ and $r_2 = 5$. The Euclidean algorithm applied to the pair $(2, 5)$ gives:

$$(2, 5) \to (2, 3) \to (2, 1) \to (1, 1).$$

Reversing this sequence, by the above reduction procedure, we obtain a sequence of bijections:

$$\mathsf{BM}^s_P(1, 1) \xrightarrow{\sim} \mathsf{BM}^s_P(2, 1) \xrightarrow{\sim} \mathsf{BM}^s_P(2, 3) \xrightarrow{\sim} \mathsf{BM}^s_P(2, 5),$$

and finally for the matrices we get:

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline & \lambda \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 0 & \lambda & 1 \\ \hline \multicolumn{2}{|c|}{} & 0 \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{} & 0 & 1 & 0 \\ \cline{3-5} \multicolumn{2}{|c|}{} & 0 & \lambda & 1 \\ \cline{3-5} \multicolumn{2}{|c|}{} & 0 & 0 & 0 \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline \multicolumn{2}{|c|}{} & 0 & 0 & 1 & 0 & 0 \\ \cline{3-7} \multicolumn{2}{|c|}{} & 0 & 0 & 0 & 1 & 0 \\ \cline{3-7} \multicolumn{2}{|c|}{} & 0 & 0 & 0 & 1 & 0 \\ \cline{3-7} \multicolumn{2}{|c|}{} & 0 & 0 & 0 & \lambda & 1 \\ \cline{3-7} \multicolumn{2}{|c|}{} & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}.$$

3.3 Matrix problem for simple torsion free sheaves

Reduced matrix problem for torsion free sheaves

The reduction for torsion free but not locally free sheaves can be done in a similar way. The only difference is that the matrices $\mu(0)$ and $\mu_\varepsilon(0)$ are no longer square:

$$\mu(0) = \begin{array}{|c|c|c|} \hline \mathbb{I}_{r_1} & 0 & 0 \\ \hline 0 & \mathbb{I}_{r_2} & 0 \\ \hline \end{array} \quad \text{and} \quad \mu_\varepsilon(0) = \begin{array}{|c|c|c|} \hline B_1 & B_{12} & B_{13} \\ \hline 0 & B_2 & B_{23} \\ \hline \end{array}. \tag{3.10}$$

The matrix $\mu_\varepsilon(0)$ has two additional blocks B_{13} and B_{23} of sizes $r_1 \times t$ and $r_2 \times t$ respectively, where t is a size defined in Section 2.4 as $t := h^0(\mathcal{M}) - r = \deg_E(\mathcal{F}) - \deg_{\widetilde{E}}(\widetilde{\mathcal{F}})$.

Analogously to the case of vector bundles, we will reduce the matrix $\mu_\varepsilon(0)$ under the condition that the identity form (3.10) of the matrix $\mu(0)$ is preserved. Thus, from now on we assume

$$F(0) = (\mathbb{I}_{r_1+r_2}, 0_t) \cdot f,$$

and hence, f inherits the lower-block-triangular structure but on three blocks.

Reduced matrix problem for simple torsion free sheaves

We obtain a new matrix problem, which reads:

$$\mu_\varepsilon(0) \mapsto F(0)\mu_\varepsilon(0)f^{-1}.$$

The allowed transformations are listed below.

1. An arbitrary invertible elementary transformation simultaneously in the first block-row and the first block-column of $\mu_\varepsilon(0)$.

2. An arbitrary invertible elementary transformation simultaneously in the second block-row and the second block-column of $\mu_\varepsilon(0)$.

3. An arbitrary invertible elementary transformations in the third block-column of $\mu_\varepsilon(0)$.

4. We can add the third block-column to the first and second ones.

5. We can add a row of the first block-row of $\mu_\varepsilon(0)$ to a row of the second one and simultaneously subtract the corresponding column of the second block-column from the corresponding column of the first one.

Block matrix category BM_P

According to notations introduced in Section 2.7, this matrix problem corresponds to the category of block matrices BM_P, where the set of indices is

$I = I \cup I' = \{1,2\} \cup \{3\}$ and $P = (I, \prec) = \{(1,2),(1,3),(2,3)\} \subset I \times I$, we visualize it as a graph

$$\begin{tikzcd} & \bullet^{2} \arrow[dr] & \\ \circ_{3} \arrow[ur] \arrow[rr] & & \bullet_{1} \end{tikzcd}.$$

Indeed, matrices $\mu_\varepsilon(0)$ in formula (3.10), are upper-block-triangular matrices B consisting of the blocks $(B_1, B_{12}, B_2, B_{13}, B_{23})$, where (B_1, B_2) are square matrices of sizes r_1 and r_2 respectively. Morphisms $S : B \to B'$ are given by lower-block-triangular matrices:

$$S = \begin{array}{|c|c|c|} \hline S_1 & 0 & 0 \\ \hline S_{21} & S_2 & 0 \\ \hline S_{31} & S_{32} & S_3 \\ \hline \end{array}$$

with block sizes (r_1, r_2) and satisfying equations

$$\overline{S}B = B'S,$$

where $\overline{S}$ is the restriction of S on the first two blocks. This equation can be considered as a matrix equation modulo the left lower block B_{21}. In terms of blocks this equation reads:

$$\begin{array}{rcl} S_1B_1 & = & B'_1S_1 + B'{}_{12}S_{21} + B'_{13}S_{31}, \\ S_1B_{12} & = & B'_{12}S_2 + B'_{13}S_{32}, \\ S_1B_{13} & = & B'_{13}S_3, \\ S_{21}B_{13} + S_2B_{23} & = & B'_{23}S_3, \\ S_{21}B_{12} + S_2B_2 & = & B'_2S_2. \end{array} \tag{3.11}$$

To work with the category BM_P a change of notations seems appropriate. Namely, let $\mathbb{s} := (s_1, s_2, s_3) := (r_1, r_2, t) \in (\mathbb{N} \cup \{0\})^3$ be a vector dimension. The category BM_P splits into strata:

$$\mathsf{BM}_P = \bigcup_{\mathbb{s} \in (\mathbb{N} \cup \{0\})^3} \mathsf{BM}_P(\mathbb{s}). \tag{3.12}$$

If $s_3 = 0$ we obtain the category BM_P for vector bundles from the previous section. Thus, we assume $s_3 > 0$. The problem can be reduced to a canonical form in terms of matrices. However, to make the treatment more consistent we postpone it to Section 7.2, where we consider it explicitly in terms of differential biquivers. The main result of Section 7.2 is Lemma 7.2.3. Here we give only its geometric interpretation.

For a brick $B \in \mathsf{BM}^s_P(\mathbb{s})$ replacing back vector dimension (s_1, s_2, s_3) by ranks (r_1, r_2, t) we obtain that there exists a unique simple matrix $\tilde{\mu}_\varepsilon(0) = B \in \mathsf{BM}_P(s_1, s_2, s_3)$ if and only if $t = 1$ and $r_1 - 1$ and $r_2 + 1$ are coprime.

From the formula (3.4): $\widetilde{\mathcal{F}} \cong \widetilde{\mathcal{O}}(c)^{r_1} \oplus \widetilde{\mathcal{O}}(c+1)^{r_2}$ and the definition of t it follows that $d \bmod r = r_2 + t$. Since the rank of $\mathcal{F}$ is $r = r_1 + r_2$, we get

$$g.c.d.(r, d) = g.c.d.(r, r_2 + t) = g.c.d.(r_1 - t, r_2 + t).$$

Hence, in terms of rank and degree we obtain the statements of Theorem 3.0.1 about torsion free sheaves which are not vector bundles.

Moreover, if coprime integers $r > 0$ and d are given, then one can reconstruct the matrix $\mu_\varepsilon(0) = B \in \mathsf{BM}^s_P(r_1, r_2, 1)$, corresponding to a unique torsion free sheaf $\mathcal{F}(r, d)$, (not a vector bundle) by the following algorithm:

Algorithm 3.3.1. Let $(r, d) \in \mathbb{Z}^2$ be coprime with positive r, and $\lambda \in \Bbbk$.

- First, by the Euclidean algorithm we find integers $0 < r_1 \le r$ and $0 \le r_2 < r$ such that $r_1 + r_2 = r$ and $cr + r_2 = d - 1$ for some $c \in \mathbb{Z}$. Thus we recover the normalization sheaf $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}}(c)^{r_1} \oplus \widetilde{\mathcal{O}}(c+1)^{r_2}$ and sizes of the original matrix problem (r_1, r_2).

- Take new sizes $(s_1, s_2) := (r_1 - 2, r_2)$, (which are sizes of the matrix problem from Lemma 7.2.2. It corresponds to the equivalence $\mathsf{BM}^s_P(r_1, r_2, 1) \longrightarrow \mathsf{BM}^s_{P'}(s_1, s_2, 1)$ where P' is a poset obtained in two steps:

$$\begin{tikzcd}[column sep=small] & \bullet^{2} \arrow[dr] & & & & \bullet^{2} \arrow[dr] & & & & \circ^{3} \arrow[dr] & \\ \circ^{3} \arrow[ur] \arrow[rr] & & \bullet^{1} & \Longrightarrow & \circ^{3} \arrow[rr] & & \bullet^{1} & \Longrightarrow & \bullet^{2} \arrow[ur] \arrow[rr] & & \bullet^{1}. \end{tikzcd} \tag{3.13}$$

- Apply the reduction on sizes $(s_1, s_2, 1)$ according to schemes (7.5) and (7.6). At the final step we are bound to obtain either reduction (7.8) or (7.9).

Using the canonical form B_n of (7.10) as the input data we construct the canonical form by induction on sizes $(s_1, s_2, 1)$, where induction is determined by the reverse sequence of reduction.

- First apply to B_n step (7.11) or (7.12);

- construct the canonical form with sizes of blocks $(s_1, s_2, 1)$ inductively, following steps (7.13) and (7.14);

- for the constructed canonical form B' of a brick with sizes $(s_1, s_2, 1)$, by the induction step (7.15) recover the matrix $\mu_\varepsilon(0) = B \in \mathsf{BM}_P(r_1, r_2, 1)$.

Remark 3.3.2. Note that the algorithm can be written in a more compact way using the automaton (see Section 3.5).

3.4 Examples

Example 3.4.1. Vector bundles $\mathcal{E}$ of $\mathsf{VB}^s_E(1,0) = \mathsf{Pic}^0(E)$ have $\widetilde{\mathcal{E}} = \widetilde{\mathcal{O}}$ as the normalization sheaf and the corresponding matrices $\tilde{\mu}$ are $\boxed{1} + \varepsilon\boxed{\lambda}$, $\lambda \in \Bbbk$. For the unique torsion free but not locally free sheaf $\mathcal{F}$ of rank 1 and degree 0, one computes that $deg(\widetilde{\mathcal{F}}) = deg(\mathcal{F}) - 1 = -1$, thus $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}}(-1)$ and the corresponding matrix $\tilde{\mu}$ is

$$\begin{array}{|c|c|}\hline 1 & 0 \\ \hline\end{array} + \varepsilon \cdot \begin{array}{|c|c|}\hline 0 & 1 \\ \hline\end{array}.$$

Example 3.4.2. For vector bundles $\mathcal{E}$ from $\mathsf{VB}^s_E(2,1)$ the normalization sheaf is $\widetilde{\mathcal{E}} = \widetilde{\mathcal{O}} \oplus \widetilde{\mathcal{O}}(1)$, thus the corresponding matrices are

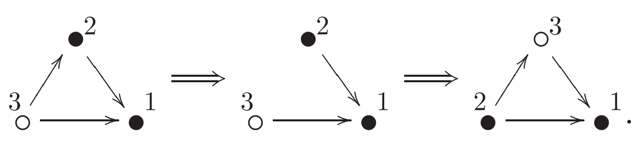
$$\tilde{\mu} = \begin{array}{|cc|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array} + \varepsilon \cdot \begin{array}{|cc|}\hline 0 & 1 \\ \hline 0 & \lambda \\ \hline\end{array}\begin{array}{c} {\scriptstyle 0} \\ {\scriptstyle 1} \end{array} \quad ,$$

where $\lambda \in \Bbbk$. For the normalization sheaf $\widetilde{\mathcal{F}}$ of the torsion free but not locally free sheaf $\mathcal{F} \in \mathsf{TF}^s(2,1)$ it holds $\deg(\widetilde{\mathcal{F}}) = \deg(\mathcal{F}) - 1 = 0$ thus $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}}^2$ and the corresponding matrix is

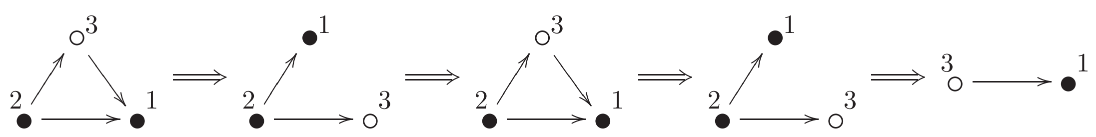
$$\tilde{\mu} = \begin{array}{|cc|c|}\hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline\end{array} + \varepsilon \cdot \begin{array}{|cc|c|}\hline 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline\end{array}\ {\scriptstyle 0}\ .$$

Example 3.4.3. Let us compactify the family of vector bundles from Example 3.2.3. Let $\mathcal{F} \in \mathsf{VB}^s_E(7,12)$ be an indecomposable torsion free but not locally free sheaf of rank 7 and degree 12.

Following the algorithm we obtain: $r_1 + r_2 = 7$ and $r_2 = d - 1 - r = 4$, and normalization sheaf $\widetilde{\mathcal{E}} = \widetilde{\mathcal{O}}(1)^3 \oplus \widetilde{\mathcal{O}}(2)^4$. For $(r_1, r_2) = (3,4)$ the vector of sizes is $(s_1, s_2) = (1,4)$. (Indeed, it corresponds to the equivalence $\mathsf{BM}^s_P(r_1, r_2, 1) \longrightarrow \mathsf{BM}^s_{P'}(s_1, s_2, 1)$ where P' is a poset obtained in two steps:



Reduction of sizes follows the scheme (7.6) and (7.9):



$$(4,1,1) \longrightarrow (3,1,1) \longrightarrow (2,1,1) \longrightarrow (1,1,1) \longrightarrow (1,1).$$

Thus, reversing this reduction according to steps 7.12 and 7.13 we obtain the matrices :

$$
\begin{array}{c|c|c|}
 & 1 & 3 \\ \hline
1 & 0 & 1 \\ \hline
3 & & \\ \hline
\end{array}
\Longrightarrow
\begin{array}{c|c|c|c|}
 & 3 & 1 & 2 \\ \hline
3 & & & 1 \\ \hline
1 & & 0 & 1 \\ \hline
2 & & & 0 \\ \hline
\end{array}
\Longrightarrow
\begin{array}{c|c|c|cc|}
 & 1 & 3 & 2 & \\ \hline
1 & 0 & 0 & 1 & 0 \\ \hline
3 & & & 0 & 1 \\ \hline
2 & & & 0 & 1 \\
 & & & 0 & 0 \\ \hline
\end{array}
$$

$$
\Longrightarrow
\begin{array}{c|c|c|ccc|}
 & 3 & 1 & 2 & & \\ \hline
3 & & 0 & 1 & 0 & 0 \\ \hline
1 & & 0 & 0 & 1 & 0 \\ \hline
2 & & & 0 & 0 & 1 \\
 & & & 0 & 0 & 1 \\
 & & & 0 & 0 & 0 \\ \hline
\end{array}
\Longrightarrow
\begin{array}{c|c|c|cccc|}
 & 1 & 3 & 2 & & & \\ \hline
1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline
3 & & & 0 & 1 & 0 & 0 \\ \hline
2 & & & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 1 \\
 & & & 0 & 0 & 0 & 1 \\
 & & & 0 & 0 & 0 & 0 \\ \hline
\end{array}.
$$

Finally reversing reduction (3.13) according to (7.15) we obtain the matrix $\mu_\varepsilon(0)$:

$$
\Longrightarrow
\begin{array}{c|cc|cccc|c|}
 & 1 & & 2 & & & & 3 \\ \hline
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline
2 & & & 0 & 0 & 1 & 0 & \\
 & & & 0 & 0 & 0 & 1 & \\
 & & & 0 & 0 & 0 & 1 & \\
 & & & 0 & 0 & 0 & 0 & \\ \hline
3 & & & 0 & 0 & 0 & 0 & \\ \hline
\end{array}
\Longrightarrow
\begin{array}{c|ccc|cccc|c|}
 & 1 & & & 2 & & & & 3 \\ \hline
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
2 & & & & 0 & 0 & 1 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 \\ \hline
\end{array}.
$$

3.5 Automaton of reduction

In this Section we collect some remarks about matrix problems for singular Weierstraß curves. First note the following: the degeneration of curves from a nodal one to a cuspidal one can be seen on the level of matrix problems. In Remark 6.4.6 we give a deformation describing the degeneration in terms of bocses. The problems of description of all indecomposable vector bundles belong to two different representation types, but matrix problems for simple vector bundles are quite similar. Although they are different and should be treated separately, the course of reduction is the same, in fact, it can be encoded by the same automaton (X, Γ), where $X = \mathbb{Z}_2 = \{a^+, a^-\}$ and $\Gamma = \{\gamma\}$ consists of a unique state. (See Section 7.4 for definitions concerning automaton).

(3.14)

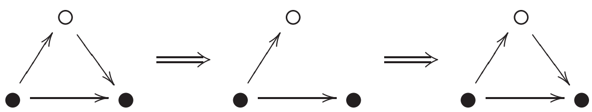
Although this automaton acts differently on matrices in the cases of nodal and cuspidal curves, on discrete parameters it operates in the same way

$$\begin{aligned} a^+ &: (r_1, r_2) \mapsto (r_1 - r_2, r_2); \\ a^- &: (r_1, r_2) \mapsto (r_1, r_2 - r_1). \end{aligned}$$

Let $\Omega := \langle a^+, a^- \rangle$ be the semigroup of paths. Reversing paths of the automaton we describe the course of induction and obtain a semigroup Ω^*. However, in this particular case $\Omega = \Omega^*$. For a path $p \in \Omega$ such that $\mathsf{BM}^s_P(r_1, r_2) \xrightarrow{p} \mathsf{BM}^s_P(r'_1, r'_2)$, where $(r'_1, r'_2) < (r_1, r_2)$ the path $p^* \in \Omega^*$ obtained from p by reversing arrows, acts as $\mathsf{BM}^s_P(r'_1, r'_2) \xrightarrow{p^*} \mathsf{BM}^s_P(r_1, r_2)$. Taking a pair (r_1, r_2) as input data we recover a path p such that $p(r_1, r_2) = (1, 0)$. For an element $\lambda \in \Bbbk$ we can recover a canonical form $B(\lambda) = p^*(\lambda) \in \mathsf{BM}^s_P(r_1, r_2)$.

Remark 3.5.1. Note that the same automaton encodes either reduction for torsion free sheaves. But to see it we should take into consideration only *principal* states, i.e. states where the reduction can terminate. Define a^- and a^+ by a composition:

a^+ and a^- :



In both cases for vector bundles and torsion free sheaves the action on geometric invariants is the same:

$$\begin{aligned} a^+ &: (r, d) \mapsto (r - d, d); \\ a^- &: (r, d) \mapsto (r, d - r). \end{aligned}$$

Remark 3.5.2. The semigroup of paths $\Omega = \langle a^+, a^- \rangle$ generates the group $\mathrm{SL}(2, \mathbb{Z})$. On the other hand it is interesting to note that the group of autoequivalences $\mathrm{Aut}(\mathcal{D}^b(\mathsf{Coh}_E)) = \langle \mathsf{T}_{\mathcal{O}}, \mathsf{T}_{\Bbbk(p_0)} \rangle$ also acts as $\mathrm{SL}(2, \mathbb{Z})$ on the K-group, or what is equivalent on rank and degree. By Theorem 4.1 of [BK3] autoequivalences $T_{\mathcal{O}}$ and T_{p_0} send stable sheaves to stable sheaves. Moreover, a continuous parameter λ can be considered as a regular point on the curve E, hence it is preserved under the action of $T_{\mathcal{O}}$ and T_{p_0}. Therefore, for singular Weierstraß curves the action of the matrix reduction coincides with the action of Fourier-Mukai transforms, namely: a^+ acts as $\mathsf{T}_{\mathcal{O}}$ and a^- acts as $(\mathsf{T}_{\Bbbk(p_0)})^{-1}$.

3.6 Universal bundle

There are many classical results about the existence of moduli spaces of stable vector bundles (torsion free sheaves) on an irreducible reduced curve, see [Ati57, Gro62, New78, Ses82]. Note that for curves of arithmetic genus bigger than one the class of simple vector bundles (torsion free sheaves) is bigger than that of stable. However, for irreducible curves of arithmetic genus one, (i.e. Weierstraß curves) these notions are equivalent (see Lemma 2.2 in [BD03] or Lemma 3.3 in [Bur03]). In this section we show that the classification of vector bundles and torsion free sheaves in terms of matrices and canonical forms gives, in fact, a description of fine moduli spaces for $\mathsf{VB}^s_E(r,d)$ and its compactification $\overline{\mathsf{VB}^s_E}(r,d) = \mathsf{TF}^s_E(r,d)$.

Let us recall basic definitions and fix notation.

Definitions and remarks

Let X be an irreducible and reduced curve over an algebraically closed field $\Bbbk$. We denote the category of sets by Sets. Recall that Sch denotes the category of Noetherian separated schemes over $\Bbbk$, and for $T \in \mathsf{Sch}$, we denote the category of coherent sheaves on T by Coh_T. The *moduli functor* of stable torsion free sheaves on X is defined as follows

$$\mathsf{TF}^s_X(r,d) : \mathsf{Sch} \to \mathsf{Sets}$$
$$T \mapsto \left\{ \mathcal{F} \in \mathsf{Coh}_{X\times T} \;\middle|\; \begin{array}{l} \mathcal{F} \text{ is flat over } T; \\ \mathcal{F}|_{X\times\{s\}} \in \mathsf{TF}^s_X(r,d) \end{array} \right\} \bmod \sim,$$

where the equivalence relation $\sim$ is given by the rule $\mathcal{F} \sim \mathcal{F} \otimes pr_T^*\mathcal{L}$ for $pr_T : X \times T \to T$ and $\mathcal{L} \in \mathsf{Pic}(T)$.

Remark 3.6.1. The moduli functor of stable vector bundles is denoted by $\mathsf{VB}^s_X(r,d)$. We treat the case of stable torsion free sheaves $\mathsf{TF}^s_X(r,d)$, since it is more general and contains the case of vector bundles. However, we need a different notation to separate the two cases.

The moduli functor $\mathsf{TF}^s_X(r,d)$ is *representable* if there exists a scheme $\bar{\mathscr{M}} \in \mathsf{Sch}$ and an invertible natural transformation of functors $\alpha : \mathsf{TF}^s_X(r,d) \to \mathrm{Hom}_{\mathsf{Sch}}(-,\bar{\mathscr{M}})$. In particular, for any morphism of schemes $f : T \to T'$ the following diagram commutes:

$$\begin{tikzcd}
\mathsf{TF}^s_X(r,d)(T) \arrow[r, "\alpha(T)", "\sim"'] \arrow[d, "\mathsf{TF}^s_X(r,d)(f)"'] & \mathrm{Hom}_{\mathsf{Sch}}(T,\bar{\mathscr{M}}) \arrow[d, "f^*"] \\
\mathsf{TF}^s_X(r,d)(T') \arrow[r, "\alpha(T')", "\sim"'] & \mathrm{Hom}_{\mathsf{Sch}}(T',\bar{\mathscr{M}})
\end{tikzcd}$$

A sheaf $\overline{\mathcal{P}} \in \mathsf{Coh}_{X\times\bar{\mathscr{M}}}$ such that for its isomorphism class $[\overline{\mathcal{P}}(r,d)] \in \mathsf{TF}^s_X(r,d)(\bar{\mathscr{M}})$ we have $[\overline{\mathcal{P}}(r,d)] = \alpha^{-1}(\bar{\mathscr{M}})(\mathrm{id}_{\bar{\mathscr{M}}})$ is called a *universal sheaf*. In this case we say that the pair $(\bar{\mathscr{M}}, \overline{\mathcal{P}}(r,d))$ *represents* the functor $\mathsf{TF}^s_X(r,d)$, or simply the scheme $\bar{\mathscr{M}}$ represents $\mathsf{TF}^s_X(r,d)$. If the sheaf $\overline{\mathcal{P}}(r,d)$ has a universal property, namely: for any $\mathcal{F} \in \mathsf{Coh}_{X\times T}$ there exists a unique morphism of schemes $\varphi : T \to \bar{\mathscr{M}}$ such that $\mathcal{F} = (\mathrm{id}_X \times \varphi)^*\overline{\mathcal{P}}(r,d)$. The pair $(\bar{\mathscr{M}}, \overline{\mathcal{P}})$ is called a *fine moduli space.*

Note that by $\mathscr{M}$ and $\mathcal{P}$ we denote the a space of parameters and a universal sheaf for the moduli problem $\mathsf{VB}^s_E(r,d)$.

Example 3.6.2. (Poincaré sheaf). Let E be a Weierstraß cubic curve. Then the moduli functor $\mathsf{Pic}^0(E) = \mathsf{VB}^s_E(1,0)$ is representable by the pair $(E_{reg}, \mathcal{P}(1,0))$, where $E_{reg} := E \setminus \mathrm{Sing}(E)$ and $\mathcal{P}(1,0) = \mathcal{J}_\Delta \otimes pr_1^*\mathcal{O}(p_0) \otimes pr_2^*(\mathcal{O}(p_0))$ for the ideal sheaf of the diagonal $\mathcal{J}_\Delta$, some fixed smooth point $p_0 \in E$ and projections pr_1 and pr_2 :

$$\begin{tikzcd} & E \times E_{reg} \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ E & & E_{reg}. \end{tikzcd}$$

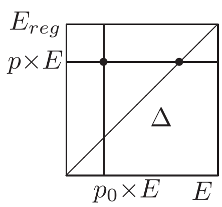
Hence, for any point $p \in E_{reg}$

$$\begin{aligned} \mathcal{P}(1,0)|_{E\times\{p\}} &= \big(\mathcal{J}_\Delta \otimes pr_1^*\mathcal{O}(p_0) \otimes pr_2^*\mathcal{O}(p_0)\big)|_{E\times\{p\}} \\ &\cong \big(\mathcal{J}_\Delta \otimes pr_1^*\mathcal{O}(p_0)\big)|_{E\times\{p\}} \\ &\cong \mathcal{O}_E(-p+p_0). \end{aligned}$$

This argument requires some explanations. The first isomorphism follows from the fact that $pr_2^*\mathcal{O}(p_0)|_{E\times\{p\}} = \mathcal{O}_{E\times\{p\}}$. The second isomorphism is obvious for a smooth curve E, since there is an isomorphism

$$\big(\mathcal{J}_\Delta \otimes pr_1^*\mathcal{O}(p_0)\big)|_{E\times\{p\}} \cong \mathcal{O}_{E\times E_{reg}}(-\Delta + p_0 \times E)|_{E\times\{p\}} \cong \mathcal{O}_E(-p+p_0),$$

which follows from the definition of Picard group: (for a scheme T and divisors $D_1, D_2 \in \mathsf{Pic}(T)$, $\mathcal{O}_T(D_1) \otimes \mathcal{O}_T(D_2) = \mathcal{O}_T(D_1+D_2)$) and the diagram:



For E not necessarily smooth, note that the structure sheaf of the diagonal $\mathcal{O}_\Delta$ is flat over both components. Hence, the exact sequence

$$\xi : \quad 0 \to \mathcal{J}_\Delta \to \mathcal{O}_{E\times E} \to \mathcal{O}_\Delta \to 0$$

remains exact if we restrict it to $E \times \{p\}$. Since $\mathcal{J}_\Delta|_{E\times\{p\}} = \mathcal{I}_p$ is the ideal sheaf of p, hence, $\mathcal{O}_\Delta|_{E\times\{p\}} = \Bbbk(p)$ is the skyscraper sheaf $\Bbbk$ at p and $\big(\mathcal{O}_{E\times E}\big)|_{E\times\{p\}} = \mathcal{O}_{E\times\{p\}}$ is the structure sheaf of $E \times \{p\}$. Thus we get:

$$\xi|_{E\times\{p\}} : \quad 0 \to \mathcal{I}_p \to \mathcal{O}_{E\times\{p\}} \to \Bbbk(p) \to 0$$

and therefore $\big(\mathcal{J}_\Delta \otimes pr_1^*\mathcal{O}(p_0)\big)|_{E\times\{p\}} \cong \mathcal{O}_E(-p+p_0)$.

In [Gro62] Grothendieck showed that for any reduced and irreducible curve X there exists a fine moduli space of $\mathsf{Pic}(1,d)$ of invertible sheaves of degree d. This space of parameters is called the *generalized Jacobian*, we denote it by $\mathscr{J}$. Its dimension is equal $p_a(X)$, the arithmetic genus of X.

Note that if X is singular, then the moduli space of $\mathsf{VB}^s_X(1,d)$ in general is not projective but only quasi-projective. In [D'S79] D'Souza showed that there exists a natural compactification of $\mathscr{J}$, and this compactification $\overline{\mathscr{J}}$ is the moduli space of torsion free sheaves $\mathsf{TF}^s_X(1,d)$. In Example 3.6.2, for instance, we get $\overline{\mathscr{J}} = E$ and $\overline{\mathcal{P}}(1,0) = \mathcal{J}_\Delta \otimes pr_1^*\mathcal{O}(p_0) \otimes pr_2^*(\mathcal{O}(p_0))$ for $pr_2 : E\times E \to E$, and $\overline{\mathcal{P}}(1,0)|_{E\times\{p\}} = \mathcal{P}(1,0)|_{E\times\{p\}}$ for any $p \in E_{reg}$. Altman and Kleiman in [AK80] showed that, for an irreducible and reduced curve E of arithmetic genus one, the compactified Jacobian $\overline{\mathsf{Pic}}^0$ is represented by the scheme E itself.

To study vector bundles and torsion free sheaves of higher rank we need a definition of a coarse moduli space.

A *coarse moduli space* for the moduli functor $\mathsf{TF}^s_X(r,d)$ is a space of parameters $\bar{\mathscr{M}}$ and a natural transformation $\alpha : \mathsf{TF}^s_X(r,d) \to \mathrm{Hom}_{\mathsf{Sch}}(-,\bar{\mathscr{M}})$ such that $\alpha(s)$ is bijective for any ordinary point scheme $s \in \mathsf{Sch}$, and for any scheme T and any natural transformation $\beta : \mathsf{TF}^s_X(r,d) \to \mathrm{Hom}_{\mathsf{Sch}}(-,T)$ there exists a unique natural transformation $\gamma : \mathrm{Hom}_{\mathsf{Sch}}(-,\bar{\mathscr{M}}) \to \mathrm{Hom}_{\mathsf{Sch}}(-,T)$ such that $\beta = \gamma \circ \alpha$.

In [New78] Newstead showed (Theorem 5.8′) that for an irreducible and reduced curve X there exists a coarse moduli spaces $\bar{\mathscr{M}}$ of the moduli functor $\mathsf{TF}^s_X(r,d)$ of stable torsion free sheaves and there is a natural compactification of $\bar{\mathscr{M}}$ by adding classes of semi-stable torsion free sheaves[1]. Moreover, if r and d are coprime, then $\bar{\mathscr{M}}$ is a fine moduli space (Theorem 5.12′).

In particular, collecting information for a Weierstraß cubic curve we obtain the following. If $\mathcal{F}$ is a stable torsion free sheaf then its rank and degree are coprime. For a pair of coprime integers $(r,d) \in \mathbb{N}\times\mathbb{Z}$, there exists a fine moduli space $\mathsf{TF}^s_E(r,d)$. The space of parameters $\bar{\mathscr{M}}$ is a projective scheme of dimension one. The space of parameters $\mathscr{M}$ of $\mathsf{VB}^s_E(r,d)$ is an open subset in $\bar{\mathscr{M}}$. Moreover, it is known that $\overline{\mathsf{Pic}}^0 \cong E$.

[1]this formulation is a little misleading in our notations. Further by compactification we always mean the compactification of moduli spaces of vector bundles by torsion free sheaves.

Universal sheaf for moduli problem $\mathsf{TF}^s_E(r,d)$ on a singular Weierstraß curve

Let E be a cuspidal cubic curve[2]. Recall the Cartesian diagram (2.1):

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\tilde{\imath}"] \arrow[d, "\tilde{\pi}"'] & \widetilde{E} \cong \mathbb{P}^1 \arrow[d, "\pi"] \\ S \arrow[r, "\imath"] & E, \end{tikzcd}$$

where $\imath : S \to E$ is the inclusion of the closed subscheme defined by the conductor ideal and $\tilde{\imath} : \widetilde{S} \to \widetilde{E}$ its pull-back to the normalization. By Theorem 2.1.3 a vector bundle $\mathcal{E}$ can be characterized by the triple

$$\left(\pi^*\mathcal{E} \cong \widetilde{\mathcal{E}}, \imath^*\mathcal{E} \cong \mathcal{O}_S^n,\ \tilde{\pi}^*\imath^*\mathcal{E} \xrightarrow{\tilde{\mu}} \tilde{\imath}^*\pi^*\mathcal{E}\right) \tag{3.15}$$

as a pull-back of the diagram:

$$\begin{tikzcd} & \imath_*\mathcal{M} \arrow[d, hook, "\mu"] \\ \pi_*\widetilde{\mathcal{E}} \arrow[r] & \imath_*\tilde{\pi}_*\tilde{\imath}^*\widetilde{\mathcal{E}}. \end{tikzcd} \tag{3.16}$$

Assume that a pair of coprime integers $(r,d) \in \mathbb{N} \times \mathbb{Z}$, $g.c.d.(r,d) = 1$ is fixed. Note that for a cuspidal cubic curve $\mathscr{M}_E(r,d) \cong E_{reg} \cong \mathbb{A}^1$. From (r,d) we recover the normalization sheaf

$$\widetilde{\mathcal{E}} := \mathcal{O}_{\mathbb{P}^1}^{r_1}(c) \oplus \mathcal{O}_{\mathbb{P}^1}^{r_2}(c+1)$$

and introduce short notations

$$\widetilde{\mathcal{P}} := \mathcal{O}_{\mathbb{P}^1\times\mathbb{A}^1}^{r_1}(c) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{A}^1}^{r_2}(c+1) = pr^*(\widetilde{\mathcal{E}}),$$

for $pr : \mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{P}^1$. Note that $\widetilde{S} = \mathrm{Spec}(\Bbbk[\varepsilon]/\varepsilon^2)$, $\mathbb{A}^1 = \mathrm{Spec}\,\Bbbk[t]$, hence $\widetilde{S} \times \mathbb{A}^1 = \mathrm{Spec}(R)$, where $R := \Bbbk[t,\varepsilon]/\varepsilon^2$. Obviously, $(\tilde{\imath} \times \mathrm{id}_{\mathbb{A}^1})^*\widetilde{\mathcal{P}} \cong R^r$ and $(\tilde{\pi} \times \mathrm{id}_{\mathbb{A}^1})^*\mathcal{O}_{S\times\mathbb{A}^1} \cong R^r$. Let $B(\lambda) \in \mathsf{BM}_P(r_1, r_2)$ be a canonical form obtained as an output of Algorithm 3.2.2. Substituting the parameter λ by a variable t we obtain the map

$$\tilde{\mu}(t) = \mathbb{I}_r + \varepsilon B(t) : R^r \to R^r.$$

Remark 3.6.3. Note that $\tilde{\mu}(t)$ is a strict representation of the bocs $\mathscr{A}$ from Example 6.4.5 over $\Bbbk[t]$ in the sense of Definition B.0.6.

[2] we consider here a cuspidal cubic curve but for a nodal curve the arguments are almost the same

Let $\mathcal{P}(r,d)$ be the pull-back of the diagram:

$$\begin{tikzcd} & (\imath \times \mathrm{id}_{\mathbb{A}^1})_* \mathcal{O}^r_{S\times\mathbb{A}^1} \arrow[d, hook, "\mu(t)"] \\ (\pi \times \mathrm{id}_{\mathbb{A}^1})_* \widetilde{\mathcal{P}} \arrow[r] & (\imath \times \mathrm{id}_{\mathbb{A}^1})_* (\tilde{\pi} \times \mathrm{id}_{\mathbb{A}^1})_* (\tilde{\imath} \times \mathrm{id}_{\mathbb{A}^1})^* \widetilde{\mathcal{P}} \end{tikzcd} \tag{3.17}$$

in the category $\mathsf{Coh}_{E\times\mathbb{A}^1}$. Then for a point $\lambda \in \mathbb{A}^1$ we have

$$\begin{aligned} (\pi \times \mathrm{id}_{\mathbb{A}^1})_* \widetilde{\mathcal{P}}|_{E\times\{\lambda\}} &= \widetilde{\mathcal{E}}, \\ (\imath \times \mathrm{id}_{\mathbb{A}^1})_* (\tilde{\pi} \times \mathrm{id}_{\mathbb{A}^1})_* (\tilde{\imath} \times \mathrm{id}_{\mathbb{A}^1})^* \widetilde{\mathcal{P}}|_{E\times\{\lambda\}} &= \mathcal{O}^r_{\widetilde{S}} \quad \text{and} \\ (\imath \times \mathrm{id}_{\mathbb{A}^1})_* \mathcal{O}^r_{S\times\mathbb{A}^1}|_{E\times\{\lambda\}} &= \mathcal{O}^r_S, \end{aligned}$$

hence, the restriction of the diagram (3.17) to $E \times \{\lambda\}$ is the diagram (3.16). Consequently, since all sheaves in the diagram (3.17) are flat over $\mathbb{A}^1$, a restriction of a pull-back is again a pull-back, we deduce $\mathcal{P}(r,d)|_{E\times\{\lambda\}} = \mathcal{E}$.

We claim that the sheaf $\mathcal{P}(r,d)$ is a universal family of stable vector bundles of rank r and degree d. Indeed, as was shown in [New78] there exists a fine moduli space $\mathscr{M}$. Let $Q(r,d)$ be a universal sheaf. By the universal property there exists a map $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ such that $\mathcal{P}(r,d) = (\mathrm{id}_E \times \varphi)^* Q(r,d)$. Let us show that φ is an isomorphism. Indeed, a morphism $\mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism if and only if it is a set-theoretic injection. Assume φ is not an isomorphism, then there exist $\lambda, \lambda' \in \mathbb{A}^1$, $\lambda \neq \lambda'$ such that $\mathcal{P}(r,d)|_{E\times\{\lambda\}} \cong \mathcal{P}(r,d)|_{E\times\{\lambda'\}}$, but this would contradict the condition $B(\lambda) \neq B(\lambda')$, for $\lambda \neq \lambda'$. Hence φ is an isomorphism and $\mathcal{P}(r,d)$ is a universal sheaf. Let us formulate this result as a theorem:

Theorem 3.6.4. *On a singular Weierstraß curve E a description of stable vector bundles $\mathsf{VB}^s_E(r,d)$ in terms of triples and canonical forms gives rise to an explicit description of fine moduli space $\mathscr{M} \cong E_{reg}$.*

Remark 3.6.5. Adding points of stable torsion free sheaves of $\mathsf{TF}^s_E(r,d) \setminus \mathsf{VB}^s_E(r,d)$ compactifies $\mathscr{M}$ in the natural sense: $\overline{E_{reg}} \cong \overline{\mathscr{M}} \cong \bar{\mathscr{M}} \cong E$.

3.7 Comparison with generalized parabolic bundles

For a rather long time (till the middle of the 70s) there were no effective methods to study moduli spaces of vector bundles on singular curves. It was Seshadri who proposed the method of parabolic bundles (PB), which are bundles on nonsingular curves with a parabolic structure[3]. Bhosle generalized the idea considering parabolic structures over divisors. The method can be used for curves with simple nodes or cusps as singularities [Ses82, Bho92, Bho93]. To deal with more complicated singularities one should also take care of the module structure of $F_1(\widetilde{\mathcal{E}})$ (see [Co93]). The idea behind the method can be explained as follows. A *generalized parabolic bundle* (shortly GPB) on the normalization $\widetilde{X}$ is a vector bundle $\widetilde{\mathcal{E}}$ on $\widetilde{X}$ together with a *parabolic structure* over the divisor $\widetilde{S}$, which is a vector subspace $F_1(\widetilde{\mathcal{E}}) \hookrightarrow \tilde{\imath}^*\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{S}}$. The category of pairs $(\widetilde{\mathcal{E}}, F_1(\widetilde{\mathcal{E}}))$ is denoted by GPB_X. One can define the rank of a GPB $(\widetilde{\mathcal{E}}, F_1(\widetilde{\mathcal{E}}))$ to be $r := \operatorname{rank}(\widetilde{\mathcal{E}}, F_1(\widetilde{\mathcal{E}})) = (\operatorname{rank}(\widetilde{\mathcal{E}})$ and the degree of a GPB $d := \deg(\widetilde{\mathcal{E}}, F_1(\widetilde{\mathcal{E}})) := \deg(\widetilde{\mathcal{E}}) + \dim(F_1(\widetilde{\mathcal{E}}))$. The notion of stability can be introduced for GPBs by considering $slope(\widetilde{\mathcal{E}}, F_1(\widetilde{\mathcal{E}})) := (\deg(\widetilde{\mathcal{E}}) + \dim(F_1(\widetilde{\mathcal{E}})))/\operatorname{rank}(\widetilde{\mathcal{E}})$. Let $\mathsf{GPB}^s(r,d)$ be the subcategory of stable GPBs of fixed rank r and degree d. A coarse moduli space $\widetilde{\mathscr{M}}$ of $\mathsf{GPB}^s(r,d)$ is constructed using geometric invariant theory.

For a generalized parabolic bundle $(\widetilde{\mathcal{E}}, F(\widetilde{\mathcal{E}}))$ on $\widetilde{X}$ consider the canonical map

$$\varphi : \widetilde{\mathcal{E}} \twoheadrightarrow \widetilde{\mathcal{E}}(\widetilde{S}) \twoheadrightarrow \widetilde{\mathcal{E}}(\widetilde{S})/F(\widetilde{\mathcal{E}}).$$

In such a way we can define a functor $\Phi : \mathsf{GPB} \to \mathsf{TF}_X$, taking $(\widetilde{\mathcal{E}}, F(\widetilde{\mathcal{E}})) \mapsto \ker(\pi_*\varphi)$. Theorem 4 in [Bho96] asserts that the functor Φ induces maps $\Phi : \mathsf{GPB}^s(r,d) \to \mathsf{TF}^s_X(r,d)$ and $\phi : \widetilde{\mathscr{M}} \to \bar{\mathscr{M}}$ with the following properties: ϕ is surjective, it is an isomorphism on $\phi^{-1}(\mathscr{M})$ and if $g.c.d.(r,d) = 1$, then $\widetilde{\mathscr{M}}$ is the normalization of $\bar{\mathscr{M}}$. In particular, $(\widetilde{\mathcal{E}}, F_1(\widetilde{\mathcal{E}}))$ is stable if and only if its image $\mathcal{F}$ is stable.

The construction of GPB differs from the construction of triples at the point that one considers $\operatorname{im}(\mu) = \tilde{\imath}^*\widetilde{\mathcal{E}}/F_1(\widetilde{\mathcal{E}})$ and not the map μ itself, thus it can happen that two different GPBs $(\widetilde{\mathcal{E}}, F_1(\widetilde{\mathcal{E}}))$ and $(\widetilde{\mathcal{E}}, F_1'(\widetilde{\mathcal{E}}))$ determine the same sheaf $\mathcal{F} := \Phi(\widetilde{\mathcal{E}}, F_1(\widetilde{\mathcal{E}})) = \Phi(\widetilde{\mathcal{E}}, F_1'(\widetilde{\mathcal{E}}))$.

Example 3.7.1. Let us consider $\overline{\mathsf{Pic}}^0(E)$ on a cuspidal cubic curve E. According to Example 3.4.1 any line bundle $\mathcal{L}$ of degree zero corresponds to a triple $(\widetilde{\mathcal{O}}, \Bbbk, 1+\lambda\varepsilon)$, by means of the exact sequence (2.10):

$$0 \to \mathcal{L} \to \pi_*\widetilde{\mathcal{O}} \to \Bbbk \oplus \varepsilon\Bbbk/(1+\lambda\varepsilon)\Bbbk \to 0$$

and there exists a unique torsion free sheaf $\mathcal{F} = \widetilde{\mathcal{O}}(-1)$ compactifying the family. It corresponds to the triple $\big(\widetilde{\mathcal{O}}(-1), \Bbbk^2, (1,\varepsilon)\big)$ and the exact sequence:

$$0 \to \pi_*\widetilde{\mathcal{O}}(-1) \to \pi_*\widetilde{\mathcal{O}}(-1) \to 0.$$

[3] usually under a parabolic structure one understands a finite set of flags of linear subspaces of fixed dimensions and respectively flag varieties as their moduli spaces.

From the GPB construction we get the same correspondence of line bundles $\mathcal{L} \in \mathsf{Pic}^0$ and GPBs $\big(\widetilde{\mathcal{O}}, \mathbb{k} \xrightarrow{(1,\lambda)} \mathbb{k}^2\big)$ with and the same exact sequence as above. Whereas, for a unique torsion free but not locally free sheaf we get, locally:

$$0 \to \mathbb{k}[t] \to \mathbb{k}[t] \to (\mathbb{k} \oplus \varepsilon\mathbb{k})/\varepsilon\mathbb{k} = \mathbb{k} \to 0$$

and consequently $0 \to \pi_*\widetilde{\mathcal{O}}(-1) \to \pi_*\widetilde{\mathcal{O}} \to (\mathbb{k} \oplus \varepsilon\mathbb{k})/\varepsilon\mathbb{k} = \mathbb{k} \to 0$. Hence, ϕ is a set theoretic bijection on $\overline{\mathsf{Pic}}^0$.

The picture becomes vivid for a nodal cubic curve:

Example 3.7.2. Consider $\overline{\mathsf{Pic}}^0$ on the nodal cubic curve E. Analogously to the previous example any line bundle $\mathcal{L}$ of degree zero corresponds to a triple $(\widetilde{\mathcal{O}}, \mathbb{k}, (1, \lambda))$, with $\lambda \neq 0$ and the exact sequence

$$0 \to \mathcal{L} \to \pi_*\widetilde{\mathcal{O}} \to \mathbb{k}(0) \oplus \mathbb{k}(\infty)/(1, \lambda)\mathbb{k} \to 0.$$

A unique torsion free but not locally free sheaf $\mathcal{F} = \widetilde{\mathcal{O}}(-1)$ compactifying the family corresponds to the triple $\big(\widetilde{\mathcal{O}}(-1), \mathbb{k}^2, (1, 1)\big)$ and exact sequence:

$$0 \to \pi_*\widetilde{\mathcal{O}}(-1) \to \pi_*\widetilde{\mathcal{O}}(-1) \to 0.$$

From the GPB construction we get: the same correspondence of line bundles $\mathcal{L} \in \mathsf{Pic}^0$ and GPBs $\big(\widetilde{\mathcal{O}}, \mathbb{k} \xrightarrow{(1,\lambda)} \mathbb{k}^2\big)$, and the same exact sequence as above. Whereas, there are two different GPBs left, namely $\big(\widetilde{\mathcal{O}}, \mathbb{k} \xrightarrow{(1,0)} \mathbb{k}^2\big)$ and $\big(\widetilde{\mathcal{O}}, \mathbb{k} \xrightarrow{(0,1)} \mathbb{k}^2\big)$. However, $\pi_*(\widetilde{\mathcal{O}}, \mathbb{k}(0)) = \pi_*(\widetilde{\mathcal{O}}, \mathbb{k}(\infty) = (\pi_*\widetilde{\mathcal{O}}, \mathbb{k}(s)))$ and $0 \to \pi_*\widetilde{\mathcal{O}}(-1) \to \pi_*\widetilde{\mathcal{O}} \to \mathbb{k}(s) \to 0$. Hence, the morphism between moduli spaces $\phi : \widetilde{\mathscr{M}} \to \bar{\mathscr{M}}$ is not injective.

Chapter 4

Vector bundles and torsion free sheaves on a tacnode curve

In this Chapter we consider a tacnode cubic curve E (Kodaira fiber III), which is another vector-bundle-wild degeneration of an elliptic curve. We assume that E is embedded in $\mathbb{P}^2$ and consists of a conic and a line touching at some point. The matrix problem describing simple vector bundles on E is more cumbersome compared to the analogous problem for a cuspidal cubic curve. However, it can still be solved without any additional techniques by the usual matrix reduction and the classification is similar to the classification of simple vector bundles on a cuspidal cubic curve (Theorem 3.0.1). The main difference to the previous case is that the condition $g.c.d.(r, d) = 1$ is not sufficient anymore for a sheaf to be simple. The usual degree d should be replaced by the *multidegree* $\mathbb{d} = (d_1, d_2)$, where d_k is the degree of the vector bundle restricted to the k-th component.

The main result of this chapter is the following.

Theorem 4.0.1. *Let E be a Kodaira fiber III Then*

- *for a simple vector bundle or a torsion free sheaf of rank r and degree d we have*

$$g.c.d.(r, d) = 1; \tag{4.1}$$

- *for a triple of integers $(r, d_1, d_2) \in \mathbb{N} \times \mathbb{Z}^2$ with $(r, d := d_1 + d_2)$ satisfying (4.1) the isomorphism classes of simple vector bundles of given rank r and multidegree (d_1, d_2) are parametrized by $\mathbb{A}^1$;*

- *for a triple of integers $(r, d_1, d_2) \in \mathbb{N} \times \mathbb{Z}^2$ with $(r, d := d_1 + d_2 + 1)$ satisfying (4.1) there exists a unique simple torsion free and not locally free sheaf of rank r on each component and multidegree (d_1, d_2).*

Moreover, for vector bundles we provide Algorithm 4.4.1, which for a given rank and multidegree $(r, \mathbb{d}) \in \mathbb{N} \times \mathbb{Z}^2$ as input checks the condition (4.1). If it holds true then the algorithm computes the canonical form of the corresponding matrix $\tilde{\mu}$. The kernel of the algorithm is the brick-reduction automaton 7.5.2. For torsion free and not locally free sheaves we provide Algorithm 4.5.2, based on the automaton 7.7.1. Note that automatons 7.5.2 and 7.7.1 have the same form and differ only by their action on the discrete parameters.

Remark 4.0.2. It is interesting to note that the action of the semi-group of the automaton 7.5.2 on $\mathbb{Z}^3$ coincides with the braid group action induced by $G = \langle T_{\mathcal{O}}, T_1, T_2 \rangle \subseteq \mathrm{Aut}(\mathcal{D}^b(\mathsf{Coh}_E))$, defined by Seidel and Thomas in [ST01].

Proof of Theorem 4.0.1

We proceed along similar lines as in the proof of Theorem 3.0.1. It is straightforward but requires more exertion and work. We consider the case of vector bundles first and divide the proof in four steps:

Step 1. In Section 4.1 we describe the corresponding category of triples.

Step 2. In Section 4.2, we apply the *primary matrix reduction* to the matrix problem. After the primary reduction we get a matrix problem of BM_P type.

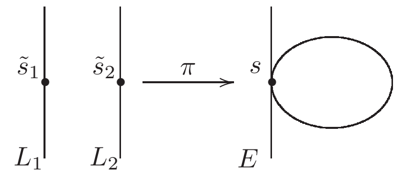
Step 3. In Section 7.3 of Chapter 7 we describe bricks of the category BM_P using the technique of bocses.

Step 4. Finally in Section 4.4, we interpret the results obtained in Section 7.3 in geometric terms as an algorithm for constructing simple vector bundles of given rank and multidegree.

The case of torsion free sheaves we consider in Section 4.5. The corresponding matrix problem is treated in Section 7.7.

4.1 Reduction to the matrix problem

Let E be a tacnode curve given by the equation $y(zy - x^2) = 0$. The normalization $\widetilde{E}$ is a disjoint union of two copies of a projective line L_1 and L_2.



On each component of the normalization $\widetilde{E} \cong L_1 \sqcup L_2 \xrightarrow{\pi} E$ choose coordinates $(z_0 : z_1)$ such that the preimages $\tilde{s_1}$ and $\tilde{s_1}$ of the singular point $s = (0 : 0 : 1)$ of E are both $0 := (0 : 1)$. For $k = 1, 2$ let $U_k := \{(z_0 : z_1) | z_1 \neq 0\}$ be an affine neighborhood of 0 on the component L_k. Let U be the disjoint union of U_1 and U_2. Introduce local coordinates $t_k := z_0/z_1$ on each component L_k. The normalization sheaf splits $\widetilde{\mathcal{O}} = \mathcal{O}_1 \oplus \mathcal{O}_2$, where $\mathcal{O}_k := \mathcal{O}_{L_k}$ is the structure sheaf of the component L_k. Thus, the direct image $\pi_*\widetilde{\mathcal{O}}$ on the open set $\pi(U)$ in local coordinates reads $\Bbbk[t_1] \oplus \Bbbk[t_2]$ and the normalization map $\mathcal{O} \hookrightarrow \pi_*\widetilde{\mathcal{O}}$ is:

$$\begin{aligned} \Bbbk[U] &\to \Bbbk[t_1] \oplus \Bbbk[t_2] \\ 1 &\mapsto (1,1), \\ x &\mapsto (t_1, t_2), \\ y &\mapsto (0, t_2^2). \end{aligned}$$

For the conductor we have $\mathcal{J}(\pi(U)) = \langle (t_1^2, 0), (0, t_2^2) \rangle$. In other words, the ideal sheaf of the scheme-theoretic preimage of s is $\widetilde{\mathcal{J}} = \left(\mathcal{I}^2_{L_1,0}, \mathcal{I}^2_{L_2,0}\right)$, where $\mathcal{I}_{L_k,0}$ denotes the ideal sheaf of the point 0 on the component L_k. Hence, $\mathcal{O}_{\widetilde{S}} \cong \widetilde{\mathcal{O}}/\widetilde{\mathcal{J}} = \mathcal{O}_1/\mathcal{I}^2_{L_1,0} \oplus \mathcal{O}_2/\mathcal{I}^2_{L_2,0}$. Following the notations from Section 2.2 we have:

$$\begin{aligned} \mathcal{O}_S &\cong \left(\Bbbk[\varepsilon]/\varepsilon^2\right)(s), \\ \mathcal{O}_{\widetilde{S}} &\cong \left(\Bbbk_1[\varepsilon_1]/\varepsilon_1^2\right)(\tilde{s}_1) \oplus \left(\Bbbk_2[\varepsilon_2]/\varepsilon_2^2\right)(\tilde{s}_2) \end{aligned}$$

and the induced map $\mathcal{O}_S \hookrightarrow \mathcal{O}_{\widetilde{S}}$ takes ε to $(\varepsilon_1, \varepsilon_2)$.

According to Section 2.2 for a triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ we fix the following data:

- a splitting $\widetilde{\mathcal{F}} \cong \Big(\underset{n\in\mathbb{Z}}{\oplus} \mathcal{O}_1(n)^{r(n,1)} \Big) \bigoplus \Big(\underset{m\in\mathbb{Z}}{\oplus} \mathcal{O}_2(m)^{r(m,2)} \Big)$, for some multiplicities $r(n,1)$ and $r(m,2)$ such that

$$\sum_{n\in\mathbb{Z}} r(n,1) = \sum_{m\in\mathbb{Z}} r(m,2) = r;$$

- an isomorphism $\mathcal{M} \cong \mathcal{O}_S^r = \left(\Bbbk[\varepsilon]/\varepsilon^2\right)^r$, (the case of torsion free sheaves, which are not vector bundles will be considered later);

- for both components $k = 1, 2$ fix trivializations:

$$\begin{aligned} \mathcal{O}_k(n) \otimes \mathcal{O}_k/\mathcal{I}^2_{(0)} &\longrightarrow \Bbbk[\varepsilon_k]/\varepsilon_k^2, \\ \zeta \otimes 1 &\longmapsto pr(\frac{\zeta}{z_1^n}) \end{aligned}$$

  for a local section ζ of $\mathcal{O}_k(n)$ on the open set U_k, where the projection

$$pr : \Bbbk[U_k] \longrightarrow \Bbbk[\varepsilon_k]/\varepsilon_k^2$$

  is the map induced by $\Bbbk[t_k] \longrightarrow \Bbbk[\varepsilon_k]/\varepsilon_k^2$, mapping $t_k \mapsto \varepsilon_k$.

With respect to all these choices the morphisms $\tilde{\mu}$, $\tilde{\imath}^* F$ and $\tilde{\pi}^* f$ can be written as matrices.

- The map $\tilde{\mu}$ can be expressed via four $r \times r$ matrices over $\Bbbk$:

$$\tilde{\mu} = (\mu_1, \mu_2) = \Big(\mu_1(0) + \varepsilon_1 \cdot \mu_{\varepsilon_1}(0), \quad \mu_2(0) + \varepsilon_2 \cdot \mu_{\varepsilon_2}(0)\Big), \tag{4.2}$$

$$\begin{tikzcd}[column sep=small] & \Big(\big(\Bbbk[\varepsilon]/\varepsilon^2\big)(s)\Big)^r \arrow[dl, "\mu_1"', "\cong"] \arrow[dr, "\mu_2", "\cong"'] & \\ \Big(\big(\Bbbk[\varepsilon_1]/\varepsilon_1^2\big)(\tilde{s}_1)\Big)^r & & \Big(\big(\Bbbk[\varepsilon_2]/\varepsilon_2^2\big)(\tilde{s}_2)\Big)^r \end{tikzcd}$$

The morphism $\tilde{\mu}$ is an isomorphism if and only if both matrices $\mu_1(0)$ and $\mu_2(0)$ are invertible.

- Repeating the consideration made for the cuspidal case, we obtain that a morphism $\tilde{\imath}^*F : \tilde{\imath}^*\widetilde{\mathcal{F}} \longrightarrow \tilde{\imath}^*\widetilde{\mathcal{F}}$ reads

$$\tilde{\imath}^*F = (\tilde{\imath}^*F_1, \tilde{\imath}^*F_2) = \Big(F_1(0) + \varepsilon_1 \tfrac{dF_1}{dz_0}(0), F_2(0) + \varepsilon_2 \tfrac{dF_2}{dz_0}(0)\Big).$$

Recall that by construction $\tilde{\imath}^*F_k$ is a lower-block-triangular matrix over $\Bbbk[\varepsilon_k]/\varepsilon_k^2$ and its diagonal blocks are matrices over $\Bbbk$.

- Obviously, we have $\tilde{\pi}^*f = (f, f)$ and $f = f(0) + \varepsilon \cdot f_\varepsilon(0) \in \mathrm{Mat}_{\Bbbk[\varepsilon]/\varepsilon^2}(r \times r)$.

A morphism (F, f) is an isomorphism, if and only if $F_1(0), F_2(0), f(0) \in \mathrm{GL}(\Bbbk, r)$. The transformation rule

$$\tilde{\mu} \mapsto \tilde{\mu}' = \tilde{\imath}^*F \circ \tilde{\mu} \circ \tilde{\pi}^*f^{-1},$$

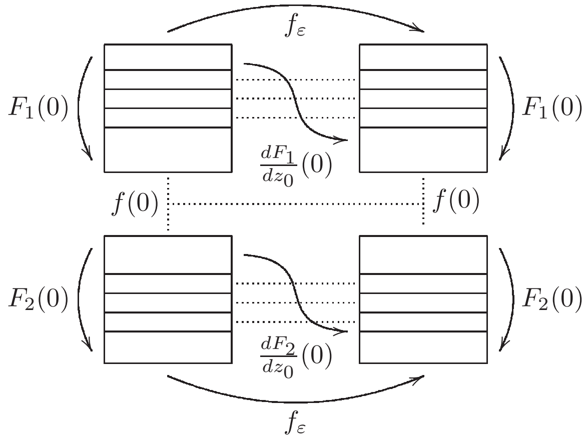
in the matrix form reads:

$$\begin{aligned} F_1(0)\mu_1(0) &= \mu_1'(0)f(0) && (4.3)\\ F_2(0)\mu_2(0) &= \mu_2'(0)f(0) && (4.4)\\ \tfrac{dF_1}{dz_0}(0)\mu_1(0) + F_1(0)\mu_{\varepsilon_1}(0) &= \mu_{\varepsilon_1}'(0)f(0) + \mu_1'(0)f_\varepsilon(0) && (4.5)\\ \tfrac{dF_2}{dz_0}(0)\mu_2(0) + F_2(0)\mu_{\varepsilon_2}(0) &= \mu_{\varepsilon_2}'(0)f(0) + \mu_2'(0)f_\varepsilon(0) && (4.6) \end{aligned}$$

Hence, the matrix problem for a tacnode curve can be formulated as follows.

Original matrix problem for a tacnode curve

There are four $r \times r$ matrices $\mu_1(0)$, $\mu_{\varepsilon_1}(0)$ and $\mu_2(0)$, $\mu_{\varepsilon_2}(0)$ over $\Bbbk$, where matrices $\mu_1(0)$ and $\mu_2(0)$ are invertible. Both pairs $\mu_1(0)$, $\mu_{\varepsilon_1}(0)$ and $\mu_2(0)$, $\mu_{\varepsilon_2}(0)$ are simultaneously divided into horizontal blocks labelled by integers called "weights." A pair of blocks of $\mu_k(0)$ and $\mu_{\varepsilon_k}(0)$, for $k = 1, 2$ marked by the same label have the same number of rows and are called *conjugated*. In a sense, we have "two copies of the matrix problem for a cuspidal cubic curve" with some new simultaneous column transformation f_ε.



The admissible transformations are listed below:

1. An arbitrary simultaneous invertible elementary transformation of columns for *all* matrices $\mu_1(0)$, $\mu_{\varepsilon_1}(0)$, $\mu_2(0)$, and $\mu_{\varepsilon_2}(0)$. Such transformations correspond to the matrix $f(0)$.

2. We can add a scalar multiple of a column of the matrix $\mu_k(0)$ to a column of the matrix $\mu_{\varepsilon_k}(0)$ simultaneously for both $k = 1, 2$. Such transformations correspond to the matrix $f_\varepsilon(0)$.

3. An arbitrary simultaneous invertible elementary row transformation of $\mu_k(0)$ and $\mu_{\varepsilon_k}(0)$, inside of any two conjugated horizontal blocks (of course, independently for each $k = 1, 2$). Such transformations correspond to the diagonal blocks of the matrix $F_k(0)$.

4. We can add a scalar multiple of any row with a lower weight to any row with a higher weight simultaneously in $\mu_k(0)$ and $\mu_{\varepsilon_k}(0)$, independently for each $k = 1, 2$. Such transformations correspond to the non-diagonal blocks of the matrix $F_k(0)$.

5. We can add a row of $\mu_k(0)$ with a lower weight to any row of $\mu_{\varepsilon_k}(0)$ with a higher weight, separately for each $k = 1, 2$. Such transformations correspond to the matrix $\frac{dF_k}{dz_0}(0)$.

One can see that this matrix problem is wild, since restricted to a component it becomes the matrix problem for a cuspidal cubic curve, which is wild.

4.2 Primary reduction of the matrix problem

Consider simple torsion free sheaves. Since Lemma 2.6.4 and equation (2.19) are satisfied for each component L_k, thus, for a simple torsion free sheaf $\mathcal{F}$, we assume

$$\widetilde{\mathcal{F}} \cong \mathcal{O}_1(c_1)^{r-\bar{d}_1} \oplus \mathcal{O}_1(c_1+1)^{\bar{d}_1} \oplus \mathcal{O}_2(c_2)^{r-\bar{d}_2} \oplus \mathcal{O}_2(c_2+1)^{\bar{d}_2} \tag{4.7}$$

for some $c_1, c_2 \in \mathbb{Z}$ and $\bar{d}_k = d_k \bmod r$. Hence, both of the matrices μ_1 and μ_2 consist of two horizontal blocks. Since the shifts c_1 and c_2 do not affect the matrix problem, we can assume that the blocks have weights 0 and 1 for both components L_1 and L_2. Having twists c_1 and c_2, can we recover the multidegree of $\mathbb{d}$ by the rule $d_k = r \cdot c_k + \bar{d}_k$. Since this matrix problem is not very convenient we transform it to a problem of BM_P type, which has a more elegant form and can be treated in a formal way.

Lemma 4.2.1. *Let $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ be a triple corresponding to a simple vector bundle. Then*

I. *the matrix $\tilde{\mu}$ can be transformed to one of the following forms:*

- *if $r \geq \bar{d}_1 + \bar{d}_2$ then*

$$\mu_1 = \left[\begin{array}{cc|c} \mathbb{I}_{s_1} & 0 & 0 \\ 0 & \mathbb{I}_{s_2} & 0 \\ \hline 0 & 0 & \mathbb{I}_{s_3} \end{array}\right] + \varepsilon_1 \left[\begin{array}{cc|c} B_1 & B_{12} & B_{13} \\ 0 & B_2 & 0 \\ \hline 0 & 0 & B_3 \end{array}\right] \begin{array}{l} \\ 0 \\ 1 \end{array} \tag{4.8}$$

$$\mu_2 = \left[\begin{array}{cc|c} 0 & 0 & \mathbb{I}_{s_3} \\ \mathbb{I}_{s_1} & 0 & 0 \\ \hline 0 & \mathbb{I}_{s_2} & 0 \end{array}\right] + \varepsilon_2 \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ 0 \\ 1 \end{array} \tag{4.9}$$

where $s_1 = r - (\bar{d}_1 + \bar{d}_2)$, $s_3 = \bar{d}_1$ and $s_2 = \bar{d}_2$;

- *if $r < \bar{d}_1 + \bar{d}_2$ then*

$$\mu_1 = \left[\begin{array}{c|cc} \mathbb{I}_{s_1} & 0 & 0 \\ \hline 0 & \mathbb{I}_{s_2} & 0 \\ 0 & 0 & \mathbb{I}_{s_3} \end{array}\right] + \varepsilon_1 \left[\begin{array}{c|cc} B_1 & 0 & B_{13} \\ \hline 0 & B_2 & B_{23} \\ 0 & 0 & B_3 \end{array}\right] \begin{array}{l} 0 \\ 1 \\ \end{array} \tag{4.10}$$

$$\mu_2 = \left[\begin{array}{c|cc} 0 & \mathbb{I}_{s_2} & 0 \\ \hline 0 & 0 & \mathbb{I}_{s_3} \\ \mathbb{I}_{s_1} & 0 & 0 \end{array}\right] + \varepsilon_2 \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} 0 \\ 1 \\ \end{array} \tag{4.11}$$

where $s_1 = r - \bar{d}_1$, $s_2 = r - \bar{d}_2$, and $s_3 = \bar{d}_1 + \bar{d}_2 - r$.

Here $\mathbb{I}_n$ denotes the identity matrix of size n, B_{ij} are non-reduced blocks of sizes $s_k \times s_j$ and $B_k = B_{kk}$.

$$\left[\begin{array}{cc|c} S_1 & 0 & 0 \\ S_{21} & S_2 & 0 \\ \hline S_{31} & 0 & S_3 \end{array}\right] \qquad \left[\begin{array}{c|cc} S_1 & 0 & 0 \\ \hline 0 & S_2 & 0 \\ S_{31} & S_{32} & S_3 \end{array}\right];$$

if $r \geq \bar{d}_1 + \bar{d}_2$, if otherwise

II. *The morphisms preserving the matrices* μ_1, μ_2 *and* μ_{ε_2} *are given by the matrices* $F_1(0)$ *of the form*
and satisfying the equation

$$F_1(0)\mu_{\varepsilon_1}|_P = \mu'_{\varepsilon_1}F_1(0)|_P, \tag{4.12}$$

where P *is the poset defined by the set of blocks of* μ_{ε_1}.

Proof. Using the transformations 1 and 3 (i.e. by applying $F_1(0)$ and $f(0)$, see equation (4.3)) we reduce the matrix $\mu_1(0)$ to the identity form $\mathbb{I}_r$. In order to preserve the form of $\mu_1(0)$ we have to take

$$f(0) = F_1(0). \tag{4.13}$$

Hence, the equation (4.4) reads

$$F_2(0)\mu_2(0) = \mu'_2(0)F_1(0). \tag{4.14}$$

The partition of $F_1(0)$ and $F_2(0)$ into horizontal blocks induces a partition of μ_2, namely:

$$\mu_2(0) = \begin{array}{cc} \overbrace{}^{r-\bar{d}_1} & \overbrace{}^{\bar{d}_1} \\ \end{array}$$
$$\mu_2(0) = \begin{array}{c} \begin{array}{cc} r-\bar{d}_1 & \bar{d}_1 \end{array} \\ \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \end{array} \begin{array}{l} \}\, r-\bar{d}_2 \\ \}\, \bar{d}_2 \end{array}$$

It means that the equation (4.14) can be rewritten as 4 equations for blocks. Let $F_1(0) = \left(\begin{smallmatrix} F^1_{00} & 0 \\ F^1_{10} & F^1_{11} \end{smallmatrix}\right)$ and $F_2(0) = \left(\begin{smallmatrix} F^2_{00} & 0 \\ F^2_{10} & F^2_{11} \end{smallmatrix}\right)$, then starting with the equality $F^2_{00}A_2 = A_2F^1_{11}$ reduce A_2 to the form $\left(\begin{smallmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Performing transformations F^1_{10} and F^2_{10} we obtain:

$$\mu_2(0) = \left[\begin{array}{c|cc} 0 & \mathbb{I}_{s_3} & 0 \\ A'_1 & 0 & 0 \\ \hline A_3 & 0 & A'_4 \end{array}\right].$$

Since $\mu_2(0)$ is a non-degenerate matrix, the blocks A'_1 and A'_4 are matrices of maximal rank, and can be reduced to the forms $(\mathbb{I}, 0)$ and $\left(\begin{smallmatrix}\mathbb{I}\\0\end{smallmatrix}\right)$ respectively. Hence, we get

$$\begin{array}{|cc|cc|}
\hline
0 & 0 & \mathbb{I}_{s_3} & 0 \\
\mathbb{I}_{s_1} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \mathbb{I}_{s_4} \\
0 & A'_3 & 0 & 0 \\
\hline
\end{array}\,.$$

Analogously, since $\mu_2(0)$ is a non-degenerate matrix, A'_3 can be reduced to the identity matrix. Finally, we have reduced $\mu_2(0)$ to the form:

$$\begin{array}{|cc|cc|}
\hline
0 & 0 & \mathbb{I}_{s_3} & 0 \\
\mathbb{I}_{s_1} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \mathbb{I}_{s_4} \\
0 & \mathbb{I}_{s_2} & 0 & 0 \\
\hline
\end{array}\,. \tag{4.15}$$

where s_k are sizes of identity blocks. Note that this canonical form is essentially unique (up to choice of the canonical Gauß cell). From equation (4.14) we derive $F_2(0) = \mu_2(0)F_1(0)\mu_2(0)^{-1}$, thus the matrix $F_2(0)$ consists of transposed blocks of $F_1(0)$.

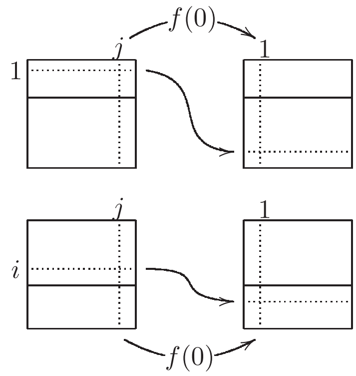
Consider the matrices $\mu_{\varepsilon_k}(0)$. Taking a proper transformation f_ε we reduce the matrix $\mu_{\varepsilon_2}(0)$ to the zero form. Observe that combining transformations f_ε and $\frac{dF_2}{dz_0}(0)$ we can also kill some blocks of the matrix $\mu_{\varepsilon_1}(0)$ keeping $\mu_{\varepsilon_2}(0)$ unchanged. Namely, applying transformations induced by f_ε the matrix $\mu_{\varepsilon_1}(0)$ can be reduced to the form:

$$\begin{array}{|cc|cc|}
\hline
B_1 & B_{12} & B_{13} & B_{14} \\
0 & B_2 & 0 & B_{24} \\
\hline
B_{31} & B_{32} & B_3 & B_{34} \\
0 & B_{42} & 0 & B_4 \\
\hline
\end{array}\,.$$

The transformation $\frac{dF_1}{dz_0}(0)$ kills the left lower (big) block of the matrix μ_{ε_1}. Finally the matrix μ_{ε_1} becomes

$$\begin{array}{|cc|cc|}
\hline
B_1 & B_{12} & B_{13} & B_{14} \\
0 & B_2 & 0 & B_{24} \\
\hline
0 & 0 & B_3 & B_{34} \\
0 & 0 & 0 & B_4 \\
\hline
\end{array}\,. \tag{4.16}$$

Endomorphisms. Note that there are two possibilities to obtain zero in the left lower (small) block B_{41} of the matrix $\mu_{\varepsilon_1}(0)$. That implies that there exists a nontrivial endomorphism of $\tilde{\mu}$. In other words, we claim that if the matrix A_2 contains a zero row and a zero column then $\tilde{\mu}$ has a nonscalar endomorphism. Indeed, assume there is a column, say $j := s_1 + s_2 + s_3 + 1$, of the matrix $\mu_2(0)$, whose intersection with the block A_2 is zero. Choosing $F_1(0)$, $F_2(0)$ to be identity matrices and f_ε, $F_{\varepsilon_1} := \frac{dF_1}{dz_0}(0)$ and $F_{\varepsilon_2} := \frac{dF_2}{dz_0}(0)$ to be zero in each entry but $f_\varepsilon(j,1) = 1$, $F_{\varepsilon_1}(j,1) = -1$ and $F_{\varepsilon_2}(s_1 + s_2 + 1, 1) = -1$, where i is the row that $\big[\mu_2(0)\big](i,1) = 1$, we get a a nonscalar endomorphism which can be visualized as follows:



(4.17)

In other words, this non-scalar endomorphism is constructed by adding simultaneously the columns j of the matrices $\mu_1(0)$ and $\mu_2(0)$ to the first columns of the matrices $\mu_{\varepsilon_1}(0)$ and $\mu_{\varepsilon_2}(0)$, (the transformation f_ε); and adding the i-th row of $\mu_1(0)$ to the changed row of μ_{ε_1} (the transformation F_{ε_1}) and adding the correspondent row of $\mu_2(0)$ to the changed row of μ_{ε_2} (the transformation F_{ε_2}).

Thus if $\tilde{\mu}$ is simple, then the block A_2 has full rank and can be reduced to the form either $(\mathbb{I}, 0)$ or $\left(\begin{smallmatrix}\mathbb{I}\\ 0\end{smallmatrix}\right)$. In particular, one of the sizes s_1 or s_4 is zero. Restricting matrices (4.15) and (4.16) to each case we obtain the dual forms of $\tilde{\mu}$ claimed in the lemma.

The equalities (4.13), (4.14) express $f(0)$ and $F_2(0)$ by $F_1(0)$. Matrices $f_\varepsilon(0)$, F_{ε_k} recover zeros on blocks $(i,j) \notin P$. Thus we obtain the statement of the part II of the lemma. □

4.3 Reduced problem

By the primary reduction we replace the map $\tilde{\mu}$ by the matrix $\mu_{\varepsilon_1}(0)$ and transformations (F, f) by $F_1(0)$ and satisfying the equation (4.12). The matrix problem obtained corresponds to the category of block matrices BM_P introduced in Section 2.7.

Proposition 4.3.1. *Let E be a cubic tacnode curve and BM_P^s be the category of block matrices, where $P = \{(i,j) |\ i \preceq j\ \} \subset I \times I$, for $I = \{1,2,3\}$ and*

$1 \prec 2, 3$. *Then there is a quasi-equivalence*

$$\mathsf{VB}^s_E(r, d_1, d_2) \xrightarrow{\sim} \mathsf{BM}^s_P(s_1, s_2, s_3),$$

where $s_1 = r - (\bar{d}_1 + \bar{d}_2)$, $s_3 = \bar{d}_1$ *and* $s_2 = \bar{d}_2$ *if* $r \geq \bar{d}_1 + \bar{d}_2$, *and* $s_1 = \bar{d}_1 + \bar{d}_2 - r$, $s_2 = r - \bar{d}_1$, $s_3 = r - \bar{d}_2$ *otherwise.*

Proof. The statement follows immediately if we combine the functor $\mathsf{VB}^s_E \longrightarrow \mathsf{Tr}^s_E$ from Theorem 2.1.3 with the reduction functor constructed in Lemma 4.2.1. Consider a category of matrices with objects $B := \mu_{\varepsilon_1}(0)$ and morphisms $S : B \to B'$, where $S := F_1(0)$ and $SB|_P = B'S|_P$. In case $\bar{d}_1 + \bar{d}_2 - r > 0$ we should also reorder indices and observe that $\mathsf{BM}^s_P(\mathbb{s}) \cong \mathsf{BM}^s_{P^t}(\mathbb{s})$, where P^t is the poset dual to P. □

It is useful to have a description of BM_P matrix problem in terms of blocks. Since an object B is a collection of matrices $(B_1, B_{12}, B_2, B_{13}, B_3)$ and a morphism $S : B \to B'$ consists of a tuple $S = (S_1, S_{21}, S_2, S_{31}, S_3)$ together with equations $SB|_P = B'S|_P$, thus for blocks this equation reads as a system:

$$\begin{array}{lrcl} (1\ ) & S_1B_1 &=& B_1'S_1 + B_{13}'S_{31} + B_{12}'S_{21} \\ (1k) & S_1B_{1k} &=& B_{1k}'S_k \\ (k) & S_{k1}B_{1k} + S_kB_k &=& B_k'S_k. \end{array} \tag{4.18}$$

for $k = 2, 3$.

Note that if we consider the subsystem of equations either (1) (12) (2) or (1) (13) (3) then we obtain the matrix problem for a cuspidal cubic curve (3.7).

The matrix problem BM_P is not too complicated and can be solved without any additional technique. Applying the matrix reductions we construct various equivalences

$$\mathsf{BM}^s_P(\mathbb{s}) \xrightarrow{\sim} \mathsf{BM}^s_{P'}(\mathbb{s}')$$

for some posets $P, P' \subset I \times I$ and $\mathbb{s}' < \mathbb{s}$. However, on each step we have to prove that the constructed maps are indeed functorial In order to avoid the repetition of similar arguments, we do the reduction formally in Section 7.3 using the language of bocses introduced in Section 6.7. In the following Section we adapt the reduction algorithm obtained there.

4.4 Algorithm for constructing canonical forms

Algorithm 4.4.1. Let $(r, d_1, d_2) \in \mathbb{N} \times \mathbb{Z}^2$ with $g.c.d.(r, d_1 + d_2) = 1$ and $\lambda \in \Bbbk$.

- By the Euclidean algorithm we find integers c_k, $\bar{d}_k$ such that $d_k = c_k r + \bar{d}_k$ for $k = 1, 2$, and recover the normalization sheaf

$$\widetilde{\mathcal{F}} = \Big(\widetilde{\mathcal{O}}(c_1)^{r-\bar{d}_1} \oplus \widetilde{\mathcal{O}}(c_1+1)^{\bar{d}_1}\Big) \oplus \Big(\widetilde{\mathcal{O}}(c_2)^{r-\bar{d}_2} \oplus \widetilde{\mathcal{O}}(c_2+1)^{\bar{d}_2}\Big).$$

Recover the matrix problem $\mathsf{BM}_P(s_1, s_2, s_3)$.

- If $r > \bar{d}_1 + \bar{d}_2$ then take a triple of integers

$$(s_1, s_2, s_3) = (r - (\bar{d}_1 + \bar{d}_2), \bar{d}_2, \bar{d}_1)$$

  and the matrix problem $(ab)^+$ in notations of Section 7.5. (The poset P is defined by the order relation $1 \prec 2, 3$).

- If $r < \bar{d}_1 + \bar{d}_2$ then take a triple of integers

$$(s_1, s_2, s_3) = (r - \bar{d}_1, r - \bar{d}_2, (\bar{d}_1 + \bar{d}_2) - r)$$

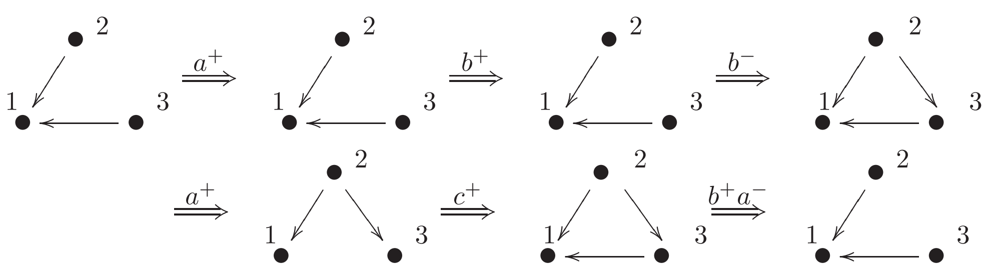
  and the matrix problem $(bc)^-$ in notations of Section 7.5. (Poset P is defined by the order relation $1, 2 \prec 3$).

Reduction using the principal automaton.

- Use the matrix problem $(mn)^\sigma$ and tuple (s_1, s_2, s_3) as the input data for the principal reduction automaton given on Figure 7.5.2. Choose a path p on it such that $p(s_1, s_2, s_3) = (1, 0, 0)$.

- To obtain a canonical form of $B \in \mathsf{BM}_P(s_1, s_2, s_3)$ we start with the one dimensional matrix $B(\lambda) = \boxed{\lambda} \in \mathsf{BM}_P(1, 0, 0)$ and reverse the matrix reduction algorithm along the path p. In this way, step by step we recover the canonical form.

Example 4.4.2. Let us describe simple vector bundles $\mathcal{E}$ with rank $r = 9$ and multidegree $(d_1, d_2) = (9c_1 + 3, 9c_2 + 4)$. The degree $d = 9(c_1 + c_2) + 7$, $\bar{d}_1 = 3$, $\bar{d}_2 = 4$ thus the rank and the degree are coprime. For any $\lambda \in \Bbbk$ there exists a unique simple vector bundle $\mathcal{F}(\lambda)$. Since $r > \bar{d}_1 + \bar{d}_2$, thus the poset P determining BM_P^s is $(ab)^+$ and the tuple of sizes is $(s_1, s_2, s_3) = (2, 4, 3)$. Input this tuple to the automaton and choice a path



The tuple of sizes (s_1, s_2, s_3) is reduced as follows:

$$\begin{aligned}(2,4,3) &\xRightarrow{a^+} (2,2,3) \xRightarrow{b^+} (2,2,1) \xRightarrow{b^-} (1,2,1)\\ &\xRightarrow{a^+} (1,1,1) \xRightarrow{c^+} (1,0,1) \xRightarrow{a^-} (1,0,1) \xRightarrow{b^+} (1,0,0)\end{aligned}$$

The matrix B correspondent to $\mathcal{E}$ can be recovered from $B(\lambda) \in \mathcal{B}^s(1)$ along the reverse path. Since the canonical form of $\mathsf{BM}_{(ac)^-}(1,1,1)$ is obvious let us start with it:

$$\begin{array}{c|c|c|} & 1 & 3 \\ \hline 1 & \lambda & 1 \\ \hline 3 & & 0 \\ \hline \end{array} \xRightarrow{c^+} \begin{array}{c|c|c|c|} & 3 & 1 & 2 \\ \hline 3 & \lambda & & 1 \\ \hline 1 & & 0 & 1 \\ \hline 2 & & & 0 \\ \hline \end{array} \xRightarrow{a^+} \begin{array}{c|c|c|cc|} & 1 & 3 & 2 & \\ \hline 1 & 0 & 0 & 1 & 0 \\ \hline 3 & & \lambda & 0 & 1 \\ \hline 2 & & & 0 & 1 \\ & & & 0 & 0 \\ \hline \end{array}$$

$$\xRightarrow{b^-} \begin{array}{c|cc|cc|c|} & 1 & & 2 & & 3 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ & 0 & \lambda & 0 & 1 & 1 \\ \hline 2 & & & 0 & 1 & \\ & & & 0 & 0 & \\ \hline 3 & & & & & 0 \\ \hline \end{array} \xRightarrow{b^+} \begin{array}{c|cc|cc|ccc|} & 1 & & 2 & & 3 & & \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ & 0 & \lambda & 0 & 1 & 0 & 1 & 0 \\ \hline 2 & & & 0 & 1 & & & \\ & & & 0 & 0 & & & \\ \hline 3 & & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 & 1 \\ & & & & & 0 & 0 & 0 \\ \hline \end{array}$$

$$\xRightarrow{a^+} \begin{array}{c|cc|cccc|ccc|} & 1 & & 2 & & & & 3 & & \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 2 & & & \lambda & 0 & 1 & 0 & & & \\ & & & 0 & 0 & 0 & 1 & & & \\ & & & 0 & 0 & 0 & 1 & & & \\ & & & 0 & 0 & 0 & 0 & & & \\ \hline 3 & & & & & & & 0 & 0 & 0 \\ & & & & & & & 0 & 0 & 1 \\ & & & & & & & 0 & 0 & 0 \\ \hline \end{array}$$

Note that parameter λ can be moved to any place on the diagonal, or we can place $\frac{\lambda}{r}$ to each diagonal place. In this form it reminds a Jordan cell:

$$
\begin{array}{c|cc|cccc|ccc|}
 & 1 & & & 2 & & & & 3 & \\
\hline
 & \frac{\lambda}{9} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & \frac{\lambda}{9} & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
\hline
 & & & \frac{\lambda}{9} & 0 & 1 & 0 & & & \\
2 & & & 0 & \frac{\lambda}{9} & 0 & 1 & & & \\
 & & & 0 & 0 & \frac{\lambda}{9} & 1 & & & \\
 & & & 0 & 0 & 0 & \frac{\lambda}{9} & & & \\
\hline
 & & & & & & & \frac{\lambda}{9} & 0 & 0 \\
3 & & & & & & & 0 & \frac{\lambda}{9} & 1 \\
 & & & & & & & 0 & 0 & \frac{\lambda}{9} \\
\hline
\end{array}
$$

4.5 Matrix problem for simple torsion free sheaves

To avoid repetition of similar arguments here we only emphasize the differences with the case of vector bundles.

One can calculate indecomposable torsion free modules over the ring $R := \mathcal{O}_{E,s} = \Bbbk[\![(t_1, t_2), (0, t_2^2)]\!]$. Here they are: R, $R' := \Bbbk[\![(t_1, 0), (0, t_2)]\!]$ and $R_k := \Bbbk[\![t_k]\!]$, $k = 1, 2$. Thus any torsion free sheaf $\mathcal{F}$ on E locally can be decomposed

$$\mathcal{F}_s = R^{\alpha} \oplus R'^{\alpha'} \oplus R_1^{\alpha_1} \oplus R_2^{\alpha_2}.$$

Recall that $\mathcal{O}_S = \Bbbk[\varepsilon]/\varepsilon^2$, where $\varepsilon := (t_1, t_2)$. Hence, $\mathcal{O}_{S_k} \cong R_k/\mathcal{J}R_k \cong \Bbbk[t_k]/t_k^2$, where $S_k = \widetilde{S} \cap L_k$. For convenience introduce a subscheme of S denoted by S' such that $\mathcal{O}_{S'} := R'/\mathcal{J}R' = \Bbbk[\![(t_1, 0), (0, t_2)]\!]/\langle (t_1^2, 0), (0, t_2^2)\rangle$. Thus a torsion free $\mathcal{O}_S$-module $\mathcal{M}$ splits:

$$\mathcal{M} \cong \mathcal{O}_S^{\alpha} \oplus \mathcal{O}_{S'}^{\alpha'} \oplus \mathcal{O}_{S_1}^{\alpha_1} \oplus \mathcal{O}_{S_2}^{\alpha_2}$$

Remark 4.5.1. Note that if we consider $\mathcal{M}$ as a $\Bbbk[\varepsilon]/\varepsilon^2$-module then we get a decomposition

$$\mathcal{M} \cong (\Bbbk[\varepsilon]/\varepsilon^2)^{\alpha+\alpha'+\alpha_1+\alpha_2} \oplus \Bbbk^{\alpha'}.$$

For the normalization sheaf $\widetilde{\mathcal{F}}$ we get the decomposition

$$\tilde{\imath}^*\widetilde{\mathcal{F}} \cong \mathcal{O}_{S_1}{}^{\alpha+\alpha'+\alpha_1} \oplus \mathcal{O}_{S_2}{}^{\alpha+\alpha'+\alpha_2}.$$

In terms of α's the discrete parameters of the matrix problem can be calculated as follows: rank on the component L_k is

$$r_k = \alpha + \alpha' + \alpha_k.$$

Let $\mathcal{F}$ be a sheaf with the same rank on each component $r = r_1 = r_2$ (i.e. $\alpha_1 = \alpha_2$) and with the multidegree $\mathbb{d} = (d_1, d_2)$. The sequence (2.10):

$$0 \to \mathcal{F} \to \pi_* \widetilde{\mathcal{F}} \to \imath_* \tilde{\pi}_* (\tilde{\imath}^* \widetilde{\mathcal{F}} / \tilde{\pi}^* \mathcal{M}) \to 0$$

and equation (2.11) imply:

$$\begin{aligned} \deg_E(\mathcal{F}) &= \deg_{\widetilde{E}}(\widetilde{\mathcal{F}}) - \dim \tilde{\imath}^* \widetilde{\mathcal{F}} / \tilde{\pi}^* \mathcal{M} \\ &= d_1 + d_2 - 2r + 2(\alpha + \alpha' + 2\alpha_1) + \alpha', \\ &= d_1 + d_2 + \alpha' + \alpha_1. \end{aligned} \tag{4.19}$$

Following the general strategy described in Section 2.2 we choose trivializations and bases of $\mathcal{M}$ and $\tilde{\imath}^* \widetilde{\mathcal{F}}$ over $\mathbb{k}$ as it was done in Section 4.1. The matrix problem obtained is analogous to the one for vector bundles. However, the matrix $\mu_{\varepsilon_1}(0)$ is no longer square and has r rows and $r + t$ columns, where $t = \alpha' + \alpha_1$.

Primary reduction

Let $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ be a triple corresponding to a simple torsion free not locally free sheaf. Assume that $\widetilde{\mathcal{F}}$ contains at most two blocks on each component i.e.

$$\widetilde{\mathcal{F}} \cong \mathcal{O}_1(c_1)^{r - \bar{d}_1} \oplus \mathcal{O}_1(c_1 + 1)^{\bar{d}_1} \oplus \mathcal{O}_2(c_2)^{r - \bar{d}_2} \oplus \mathcal{O}_2(c_2 + 1)^{\bar{d}_2}$$

Let us reduce the matrix $\tilde{\mu}$ to its canonical form. The "primary matrix reduction" is completely analogous to the one explained in Section 4.2. Thus the matrix $\tilde{\mu}$ can be reduced to the form:

$$
\begin{array}{c|ccc|ccc|ccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
1 & \mathbb{I}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & \mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & \mathbb{I}_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
4 & 0 & 0 & 0 & \mathbb{I}_4 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & \mathbb{I}_5 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_6 & 0 & 0 & 0 \\ \hline
\end{array}
+ \varepsilon_1 \cdot
\begin{array}{c|ccc|ccc|ccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
1 & B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} & B_{17} & B_{18} & B_{19} \\
2 & 0 & B_2 & B_{23} & 0 & B_{25} & B_{26} & 0 & B_{28} & B_{29} \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
4 & 0 & 0 & 0 & B_4 & B_{45} & B_{46} & B_{47} & B_{48} & B_{49} \\
5 & Z & 0 & 0 & 0 & B_5 & B_{56} & 0 & B_{58} & B_{59} \\
6 & Z & Z & Z & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
\end{array}
$$

$$
\begin{array}{c|ccc|ccc|ccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
7 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_7 & 0 & 0 \\
4 & 0 & 0 & 0 & \mathbb{I}_4 & 0 & 0 & 0 & 0 & 0 \\
1 & \mathbb{I}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_8 & 0 \\
5 & 0 & 0 & 0 & 0 & \mathbb{I}_5 & 0 & 0 & 0 & 0 \\
2 & 0 & \mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
\end{array}
+ \varepsilon_2 \cdot
\begin{array}{c|ccc|ccc|ccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
8 & Z & 0 & 0 & Z & 0 & 0 & Z & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
\end{array}
$$

by "Z" we denote the zero block, where zero can be obtained in two ways. (In (4.17) we have seen that if there is such a block then μ has a nontrivial endomorphism).

Block matrix category BM_P

Analogously to the previous cases consider only these transformations which do not affect zero and identity blocks. Such transformations are determined by matrices $f(0)$, which inherit the block structure from $F_1(0)$ and $F_2(0)$, and have the form

$$
S = \begin{array}{|ccc|ccc|ccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\ \hline
S_1 & 0 & 0 & 0 & Z & Z & 0 & Z & Z & 1 \\
S_{21} & S_2 & 0 & 0 & 0 & Z & 0 & 0 & Z & 2 \\
S_{31} & S_{32} & S_3 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ \hline
S_{41} & 0 & 0 & S_4 & 0 & 0 & 0 & Z & Z & 4 \\
S_{51} & S_{52} & 0 & S_{54} & S_5 & 0 & 0 & 0 & Z & 5 \\
S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_6 & 0 & 0 & 0 & 6 \\ \hline
S_{71} & 0 & 0 & S_{74} & 0 & 0 & S_7 & 0 & 0 & 7 \\
S_{81} & S_{82} & 0 & S_{84} & S_{85} & 0 & S_{87} & S_8 & 0 & 8 \\
S_{91} & S_{92} & S_{93} & S_{94} & S_{95} & S_{96} & S_{97} & S_{98} & S_9 & 9 \\ \hline
\end{array}
$$

Hence, instead of original matrix problem we obtain an equivalent problem BM_P, where $P \subset I \times I$ for $I = \mathbb{Z}_9$, is the set of nonzero blocks of the matrix $\tilde{\mu}_{\varepsilon_1}(0)$. A morphism $S : B \to B'$ is defined by a matrix S as above and equation $\bar{S}B|_P = B'S|_P$, where $\bar{S}$ is the matrix S restricted on blocks $1, \ldots, 6$.

Simplicity condition

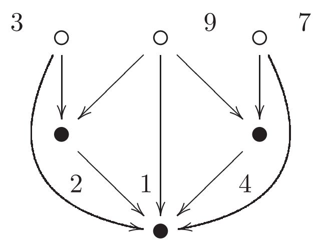
If B is simple then it contains no blocks of type Z, i.e. no "double zeros", otherwise there is a nonscalar endomorphism. Thus if B is simple then for the block entry (i, j) of B denoted by Z we have $s_i = 0$ or $s_j = 0$. We call such blocks *mutually excluding* and denote by $i \cap j$. One can easily deduce a list of mutually excluding blocks, namely:

- $1, 2, 3 \cap 6$;
- $1, 4, 7 \cap 8$;
- $1 \cap 5$.

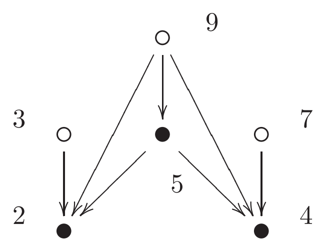
Since for both components L_1 and L_2 we assume $r - d_1 = s_1 + s_2 + s_3 > 0$ and $r - d_2 = s_1 + s_4 + s_7 > 0$, thus $s_6 = s_8 = 0$. Hence, the possible tuples of nonzero blocks are either $1, 2, 3, 4, 7, 9$ or $2, 3, 4, 5, 7, 9$; that means that B has one of the following forms

$$\begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 7 & 9 & \\ \hline B_1 & B_{12} & B_{13} & B_{14} & B_{17} & B_{19} & 1 \\ & B_2 & B_{23} & & & B_{29} & 2 \\ & & & & & & 3 \\ & & & B_4 & B_{47} & B_{49} & 4 \end{array} \quad \text{or} \quad \begin{array}{cccccc|c} 2 & 3 & 4 & 5 & 7 & 9 & \\ \hline B_2 & B_{23} & & B_{25} & B_{27} & B_{29} & 2 \\ & & & & & & 3 \\ & & B_4 & B_{45} & B_{47} & B_{49} & 4 \\ & & & B_5 & & B_{59} & 5 \end{array} \tag{4.20}$$

Thus we get problems BM_P and $\mathsf{BM}_{P'}$, where P and P' are the following posets:



and



We shall study the matrix problems BM^s_P and $\mathsf{BM}^s_{P'}$ in details in Section 7.7. Note that the matrix problem BM_P is more complicated comparing with the matrix problems from previous sections, since in the course of reduction it degenerates from a BM_P matrix problem. After some step morphisms cannot be defined by the ordinary matrix multiplication any more.

Algorithm for constructing canonical forms

Let us give an algorithm for constructing torsion free sheaves of constant rank, which are not locally free, by given discrete parameters.

Algorithm 4.5.2. Let $(r, d_1, d_2) \in \mathbb{N} \times \mathbb{Z}^2$ be a triple of integers, such that *g.c.d.*$(r, d) = 1$ for $d = d_1 + d_2 + 1$.

- By the Euclidean algorithm we find integers c_k, $\bar{d}_k$ such that $d = c_k r + \bar{d}_i$ for $k = 1, 2$. Thus we recover the normalization sheaf

$$\widetilde{\mathcal{F}} = \left(\widetilde{\mathcal{O}}(c_1)^{r-\bar{d}_1} \oplus \widetilde{\mathcal{O}}(c_1+1)^{\bar{d}_1}\right) \oplus \left(\widetilde{\mathcal{O}}(c)^{r-\bar{d}_2} \oplus \widetilde{\mathcal{O}}(c+1)^{\bar{d}_2}\right).$$

Recover the reduced matrix problem BM_P and sizes of blocks $\mathbb{s} = (s_1, s_2, s_4, s_9)$ or $\mathbb{s} = (s_2, s_4, s_5, s_9)$

- If $r > \bar{d}_1 + \bar{d}_2 + 1$ then tuple of sizes is

$$(s_1, s_2, s_4, s_9) = (r - (\bar{d}_1 + \bar{d}_2 + 1), \bar{d}_2, \bar{d}_1, 1)$$

  and the matrix problem has type $D(1, 9)$ in notations of Section 7.7.

- If $r < \bar{d}_1 + \bar{d}_2$ then take a tuple of integers

$$(s_2, s_4, s_5, s_9) = (r - \bar{d}_1, r - \bar{d}_2, (\bar{d}_1 + \bar{d}_2 + 1) - r, 1)$$

  and the matrix problem $B^{-}(9, 5)$ in notations of Section 7.7.

- Use the matrix problem and the tuple $\mathbb{s}$ as an input data for the principal reduction automaton 7.7.1. Choose a path p on it such that $p(\mathbb{s}) = (1, 0, 0)$.

- To obtain a canonical form of $B \in \mathsf{BM}_P(s_1, s_2, s_3)$ we start with $\boxed{1} \in \mathsf{BM}_P(1, 0, 0)$ and apply the reverse the matrix reduction algorithm along the path p. In this way, step by step we recover the canonical form.

Chapter 5

Vector bundles on a plane configuration of three concurrent lines

In this chapter we describe simple vector bundles on the Kodaira fiber IV (a plane configuration of three concurrent lines.)

Theorem 5.0.1. *Let E be a Kodaira fiber of type IV and $\mathcal{E}$ be a simple vector bundle on E of rank r and degree d. Then*

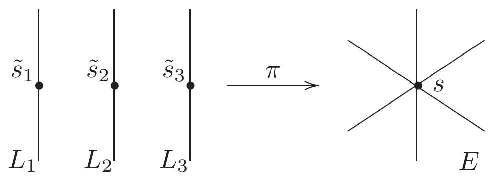
$$g.c.d.(r, d) = 1.$$

Moreover, $\mathcal{E}$ is determined by its rank $r \in \mathbb{N}$, multidegree $\mathbb{d} \in \mathbb{Z}^3$ satisfying the condition above, and its determinant $\det(\mathcal{E}) \in \mathsf{Pic}^{\mathbb{d}}(E) \cong \mathbb{A}^1$.

To prove this theorem for Kodaira fiber IV we proceed analogously as in the previous chapter. In Section 5.1 we reduce the classification of vector bundles to a matrix problem. In Section 5.2 we carry out a primary reduction and obtain a matrix problem BM_P which splits under simplicity condition into ten smaller problems of type BM_P. These categories will be treated formally in Section 7.6. Finally, in Section 5.3 we sum up the results in the Algorithm 5.3.1 for constructing canonical forms. Note that this problem is more complicated than all previous ones. The automaton of the matrix reduction 7.6.4 contains 56 states. The principal reduction automaton contains 32 states. It is still an open question whether the action of the semigroup of the principal state automaton coincides with the action of the subgroup $G \subset \mathrm{Aut}(\mathcal{D}^b(\mathsf{Coh}_E))$ introduced in Appendix D. However, there are no principal difficulties and we hope to answer these questions in a future work.

5.1 Reduction to the matrix problem

Let E be a curve consisting of three concurrent projective lines in a plane $\mathbb{P}^2$, given by the equation $xy(x-y) = 0$. Let $\widetilde{E} = \bigsqcup_{k=1}^{3} L_k \xrightarrow{\pi} E$ be the normalization map.



The goal of this chapter is to describe simple vector bundles on E. Similarly to the previous cases we reduce this classification problem to a matrix problem following the procedure from Section 2.2.

Choose coordinates $(z_0 : z_1)$ on each component $L_k \cong \mathbb{P}^1$ such that the preimage of the singular point $s = (0:0:1)$ on L_k is $0 := (0:1)$. Let $U_k = \{(z_0 : z_1) | z_1 \neq 0\}$ be affine neighborhoods of 0 on each component and U denote the disjoint union of U_1, U_2 and U_3. Introduce local coordinates $t_k := z_0/z_1$ on U_k, $k = 1, 2, 3$. The normalization map $\mathcal{O} \hookrightarrow \pi_*\widetilde{\mathcal{O}} = \pi_*(\mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \mathcal{O}_3)$, where $\mathcal{O}_k := \mathcal{O}_{L_k}$, locally at s can be written as follows:

$$\begin{aligned} \Bbbk[U] &\hookrightarrow \Bbbk[t_1] \oplus \Bbbk[t_2] \oplus \Bbbk[t_3], \\ 1 &\mapsto (1,1,1), \\ x &\mapsto (t_1, t_2, 0), \\ y &\mapsto (t_1, 0, t_3). \end{aligned}$$

Since $\mathcal{J}(U) = \langle x^2, y^2, xy \rangle$, we have

$$\mathcal{O}_S = \Bbbk[x,y]/\langle x^2, y^2, xy \rangle.$$

Note that the ideal sheaf $\widetilde{\mathcal{J}} := \pi^*\mathcal{J}$ is locally generated by $(t_1^2, 0, 0)$, $(0, t_2^2, 0)$ and $(0, 0, t_3^2)$ i.e. $\widetilde{\mathcal{J}} = \left(\mathcal{I}^2_{L_1,0}, \mathcal{I}^2_{L_2,0}, \mathcal{I}^2_{L_2,0}\right)$, where $\mathcal{I}_{L_k,0}$ denotes the ideal sheaf of the point 0 on each component L_k. Hence,

$$\mathcal{O}_{\widetilde{S}} \cong \widetilde{\mathcal{O}}/\widetilde{\mathcal{J}} \cong \bigoplus_{k=1}^{3} \mathcal{O}_k/\mathcal{I}^2_{L_k,0}.$$

As in the previous chapters, for a triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ we fix:

- a splitting $\widetilde{\mathcal{F}} \cong \bigoplus_{k=1}^{3} \left(\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_k(n)^{r(n,k)} \right)$ with $\sum_{n\in\mathbb{Z}} r(n,k) = r$;
- an isomorphism $\mathcal{M} \cong \mathcal{O}_S^r = \left(\Bbbk[x,y]/\langle x^2, y^2, xy \rangle\right)^r$;
- for each component $k = 1, 2, 3$ we take the trivializations

$$\begin{aligned} \mathcal{O}_k(n) \otimes \mathcal{O}_k/\mathcal{I}^2_{(0)} &\longrightarrow \Bbbk_k[\varepsilon_k]/\varepsilon_k^2, \\ \zeta \otimes 1 &\longmapsto pr\left(\frac{\zeta}{z_1^n}\right) \end{aligned}$$

  for a local section ζ of $\mathcal{O}_k(n)$ on an open set U_k containing $(0:1)$, where the projection
$$pr : \Bbbk[U_k] \longrightarrow \Bbbk[\varepsilon_k]/\varepsilon_k^2$$
  is the map induced by $\Bbbk[t_k] \longrightarrow \Bbbk[\varepsilon_k]/\varepsilon_k^2$, mapping $t_k \mapsto \varepsilon_k$.

With respect to all these choices the morphisms $\tilde{\mu}$, $\tilde{\imath}^*F$ and $\tilde{\pi}^*f$ can be written as matrices.

- The map $\tilde{\mu}$ can be written as a combination of six $r \times r$ matrices:

$$\tilde{\mu} = (\mu_1, \mu_2, \mu_3) = \Big(\mu_1(0)+\varepsilon_1\cdot\mu_{\varepsilon_1}(0),\ \ \mu_2(0)+\varepsilon_2\cdot\mu_{\varepsilon_2}(0),\ \ \mu_3(0)+\varepsilon_3\cdot\mu_{\varepsilon_3}(0)\Big), \tag{5.1}$$

  The morphism $\tilde{\mu}$ is invertible if and only if all $\mu_k(0)$, for $k = 1, 2, 3$ are invertible.

- Repeating the consideration for the cuspidal cubic curve, we obtain that $\tilde{\imath}^*F : \tilde{\imath}^*\widetilde{\mathcal{F}} \longrightarrow \tilde{\imath}^*\widetilde{\mathcal{F}}$ is

$$\tilde{\imath}^*F = (\tilde{\imath}^*F_1, \tilde{\imath}^*F_2, \tilde{\imath}^*F_3) = \Big(F_1(0)+\varepsilon_1\tfrac{dF_1}{dz_0}(0), F_2(0)+\varepsilon_2\tfrac{dF_2}{dz_0}(0), F_3(0)+\varepsilon_3\tfrac{dF_3}{dz_0}(0)\Big).$$

- Obviously, $\tilde{\pi}^*f = (f, f, f)$, where

$$f(0) + x \cdot f_x(0) + y \cdot f_y(0) \in \mathrm{Mat}(\Bbbk[x, y]/\langle x^2, y^2, xy\rangle, r \times r)$$

  and f is invertible if and only if $f(0) \in \mathrm{GL}(\Bbbk, r)$.

A morphism (F, f) is an automorphism, if and only if $F_1(0), F_2(0), F_3(0)$ and $f(0)$ are invertible $r \times r$ matrices over $\Bbbk$.

The transformation rule $\tilde{\mu} \mapsto \tilde{\mu}' = \tilde{\imath}^*F \circ \tilde{\mu} \circ \tilde{\pi}^*f^{-1}$, in matrix form reads:

$$F_1(0)\mu_1(0) = \mu_1'(0)f(0) \tag{5.2}$$
$$F_2(0)\mu_2(0) = \mu_2'(0)f(0) \tag{5.3}$$
$$F_3(0)\mu_3(0) = \mu_3'(0)f(0) \tag{5.4}$$
$$\tfrac{dF_1}{dz_0}(0)\mu_1(0) + F_1(0)\mu_{\varepsilon_1}(0) = \mu_{\varepsilon_1}'(0)f(0) + \mu_1'(0)f_x(0) + \mu_1'(0)f_y(0) \tag{5.5}$$
$$\tfrac{dF_2}{dz_0}(0)\mu_2(0) + F_2(0)\mu_{\varepsilon_2}(0) = \mu_{\varepsilon_2}'(0)f(0) + \mu_2'(0)f_x(0) \tag{5.6}$$
$$\tfrac{dF_3}{dz_0}(0)\mu_3(0) + F_3(0)\mu_{\varepsilon_3}(0) = \mu_{\varepsilon_3}'(0)f(0) + \mu_3'(0)f_y(0) \tag{5.7}$$

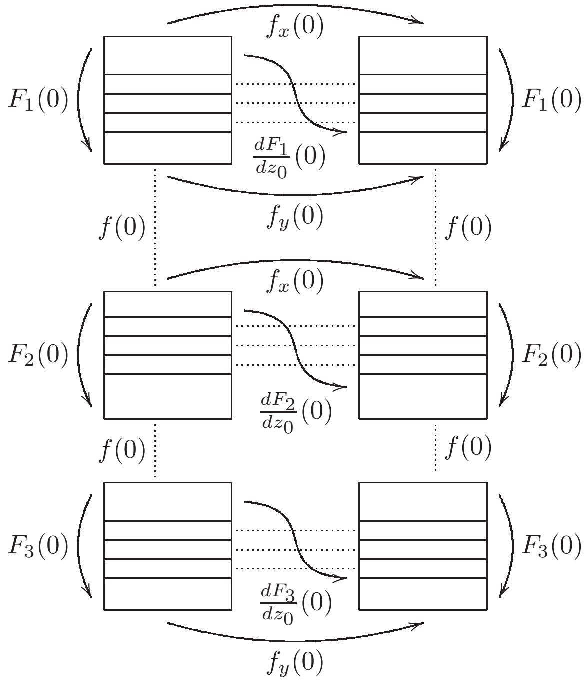
Hence, the matrix problem can be formulated as follows:

Original matrix problem

There are six $r \times r$ matrices $\mu_1(0)$, $\mu_{\varepsilon_1}(0)$, $\mu_2(0)$, $\mu_{\varepsilon_2}(0)$ and $\mu_3(0)$, $\mu_{\varepsilon_3}(0)$, where $\mu_1(0)$, $\mu_2(0)$ and $\mu_2(0)$ are invertible. The pairs $(\mu_k(0), \mu_{\varepsilon_k}(0))$ are simultaneously divided into horizontal blocks labelled by integers called weights.

Hence, we have three copies of the matrices obtained for a cuspidal cubic curve, with new simultaneous column transformation $f_x(0)$ and $f_y(0)$.

The admissible transformations are the following:



1. An arbitrary invertible elementary transformation of columns simultaneously for *all* matrices μ. Such transformations correspond to the matrix $f(0)$.

2. For the pair of matrices $\tilde{\mu}_1$ and $\tilde{\mu}_3$ we can simultaneously add a scalar multiple of any column of the matrix μ_k to any column of the matrix μ_{ε_k}, for $k = 1, 3$ This is the transformation $f_x(0)$.

3. For the pair of matrices $\tilde{\mu}_2$ and $\tilde{\mu}_3$ we can simultaneously add a scalar multiple of any column of the matrix μ_k to any column of the matrix μ_{ε_k}, for $k = 2, 3$. This is the transformation $f_y(0)$.

4. An arbitrary invertible row transformation of $\mu_k(0)$ and $\mu_{\varepsilon_k}(0)$, simultaneously inside of any two conjugated horizontal blocks (of course separately for each $k = 1, 2, 3$). Such transformations correspond to diagonal blocks of the matrix $F_k(0)$.

5. We can add a scalar multiple of any row with a lower weight to any row with a higher weight simultaneously in $\mu_k(0)$ and $\mu_{\varepsilon_k}(0)$, separately for each $k = 1, 2, 3$. Such transformations correspond to the non-diagonal blocks of the matrix $F_k(0)$.

6. We can add a row of $\mu_k(0)$ with a lower weight to any row of $\mu_{\varepsilon_k}(0)$ with a higher weight, separately for each $k = 1, 2, 3$. Such transformations correspond to the matrix $\frac{dF_k}{dz_0}(0)$.

Simplicity condition

Since $\widetilde{\mathcal{J}}$ is equal to the square of the ideal sheaf of 0 on $\mathcal{O}_k$ for $k = 1, 2, 3$, thus the statement of Lemma 2.6.4 holds true. If a triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ is simple, then

$$\widetilde{\mathcal{F}} \cong \bigoplus_{k=1}^{3} \Big(\mathcal{O}_k(c_k)^{r-\bar{d}_k} \oplus \mathcal{O}_k(c_k+1)^{\bar{d}_k} \Big) \tag{5.8}$$

and every $\tilde{\mu}_k$ has at most two horizontal blocks.

5.2 Primary reduction of the matrix problem

In Section 4.2 we have reduced the matrix $\mu_1(0)$ to the identity form $\mathbb{I}$ and the matrix $\mu_2(0)$ to the form (4.15):

$$\mu_2(0) = \begin{array}{|cc|cc|}\hline 0 & 0 & \mathbb{I} & 0 \\ \mathbb{I} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbb{I} \\ 0 & \mathbb{I} & 0 & 0 \\ \hline \end{array}\,.$$

There were some nonscalar endomorphisms. To avoid their appearance we have put some restrictions on the matrix $\tilde{\mu}$ and obtain two mutually excluding forms of $\tilde{\mu}$. Here we proceed analogously. Having canonical forms of $\mu_1(0)$ and $\mu_2(0)$ we reduce the matrix $\mu_3(0)$. Then we analyze endomorphisms of $\tilde{\mu}$.

The matrix $f(0)$, preserving matrices $\mu_1(0)$ and $\mu_2(0)$ unchanged, has the following lower block-triangular form:

$$f(0) = \begin{array}{|cc|cc|}\hline * & & & \\ * & * & & \\ \hline * & & * & \\ * & * & * & * \\ \hline \end{array}\,,$$

where the empty blocks are always zero and "$*$" denote a nonreduced block. Let us find a canonical form of $\mu_3(0)$ with respect to the transformations

$$\mu_3(0) \mapsto F_3(0)\mu_3(0)f(0)^{-1}.$$

The splitting of $F_3(0)$ and $f(0)$ into blocks induces the following block structures for $\mu_3(0)$:

$$\begin{array}{|c|c|c|c|}\hline * & * & * & * \\ * & * & * & * \\ \hline * & * & * & * \\ * & * & * & * \\ \hline \end{array}\,,$$

We proceed as in previous sections: reduce blocks "$*$" to the Gauß form $\left(\begin{smallmatrix}\mathbb{I}&0\\0&0\end{smallmatrix}\right)$ and kill nonzero blocks, where it is possible. Starting again with the right upper corner block we obtain:

$$\begin{array}{|c|c|c|cc|}\hline 0&0&0&\mathbb{I}&0\\ *&*&*&0&0\\ \hline 0&0&0&0&\mathbb{I}\\ *&*&*&0&0\\ \hline \end{array}$$

Since there is no addition of columns from the third column block to the second, thus on the second step we get

$$\begin{array}{|c|c|cc|cc|}\hline 0&0&0&0&\mathbb{I}&0\\ 0&*&\mathbb{I}&0&0&0\\ *&*&0&0&0&0\\ \hline 0&0&0&0&0&\mathbb{I}\\ 0&*&0&\mathbb{I}&0&0\\ *&*&0&0&0&0\\ \hline \end{array}$$

Dealing with the second column-block we start with the lower part:

$$\begin{array}{|c|cc|cc|cc|}\hline 0&0&0&0&0&\mathbb{I}&0\\ 0&0&*&\mathbb{I}&0&0&0\\ 0&\mathbb{I}&0&0&0&0&0\\ *&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&\mathbb{I}\\ 0&0&0&0&\mathbb{I}&0&0\\ 0&0&\mathbb{I}&0&0&0&0\\ *&0&0&0&0&0&0\\ \hline \end{array}$$

Reducing the first block we obtain:

$$\begin{array}{|cc|cc|cc|cc|}\hline 0&0&0&0&0&0&\mathbb{I}&0\\ 0&0&0&*&\mathbb{I}&0&0&0\\ 0&0&\mathbb{I}&0&0&0&0&0\\ \mathbb{I}&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&\mathbb{I}\\ 0&0&0&0&0&\mathbb{I}&0&0\\ 0&0&0&\mathbb{I}&0&0&0&0\\ 0&\mathbb{I}&0&0&0&0&0&0\\ \hline \end{array}$$

It turns out that a unique remaining block can be reduced to the form $\left(\begin{smallmatrix}\mathbb{I}&0\\0&0\end{smallmatrix}\right)$ too. That implies subdivisions for reduced blocks. Finally, marking blocks by numbers $0, \ldots, 9$ we get the following:

$$
\mu_3(0) = \begin{array}{c|cc|ccc|ccc|cc|}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_8 & 0 \\
5 & 0 & 0 & 0 & \mathbb{I}_3 & 0 & \mathbb{I}_3 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_6 & 0 & 0 & 0 \\
2 & 0 & 0 & \mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{I}_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_9 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_7 & 0 & 0 \\
3 & 0 & 0 & 0 & \mathbb{I}_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & \mathbb{I}_4 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & \mathbb{I}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

By s_i we denote the size of the block i. One can see that sizes s_3 and s_5 are equal.

With respect to this marking rows of the matrices $\mu_2(0)$ and$\mu_{\varepsilon_2}(0)$ are ordered as follows (5,6,7,0,1,8,9,2,3,4) i.e.

$$
\mu_2(0) = \begin{array}{c|cc|ccc|ccc|cc|}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
5 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_5 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_6 & 0 & 0 & 0 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_7 & 0 & 0 \\
0 & \mathbb{I}_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & \mathbb{I}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_8 & 0 \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_9 \\
2 & 0 & 0 & \mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & \mathbb{I}_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & \mathbb{I}_4 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

From the equation $F_3(0)\mu_3(0) = \mu_3(0)f(0)$ we obtain that the matrix $f(0)$, preserving $\mu_3(0)$ in this form has the form shown below. For the sake of symmetry we transpose blocks 5 and 6.

$$
f(0) = \begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \\ 5 \\ 7 \\ 8 \\ 9
\end{array}
\begin{array}{cc|ccc|ccc|cc}
0 & 1 & 2 & 3 & 4 & 6 & 5 & 7 & 8 & 9 \\
\hline
* & & & & & & & & & \\
* & * & & & & & & & & \\
\hline
* & & * & & & & & & & \\
* & \mathrm{X} & * & \mathrm{Z} & & & & & & \\
* & * & * & * & * & & & & & \\
\hline
* & & & & & * & & & & \\
* & \mathrm{X} & & & & * & \mathrm{Z} & & & \\
* & * & & & & * & * & * & & \\
\hline
* & & * & \mathrm{Y} & & * & \mathrm{Y} & & * & \\
* & * & * & * & * & * & * & * & * & * \\
\hline
\end{array}
$$

As before, here empty spaces stand for zero entries and stars stand for one time appearing blocks that can be non-zero. By X, Y and Z we denote blocks appearing twice. For the sake of symmetry we transpose blocks 5 and 6.

Taking proper $f_x(0)$ and $f_y(0)$ matrices $\mu_{\varepsilon_2}(0)$ and $\mu_{\varepsilon_3}(0)$ can be reduced to zero; then from the equations $F_k(0)\mu_k(0) = \mu_k(0)f(0)$ for $k = 2, 3$ we get the form of matrices $f_x(0)$ and f_y that leave matrices $\mu_{\varepsilon_2}(0)$ and $\mu_{\varepsilon_3}(0)$ in the zero form.

Taking proper $F_1(0)$, $f_x(0)$ and $f_y(0)$ the matrix $\mu_{\varepsilon_1}(0)$ can be reduced to the form:

$$
\mu_{\varepsilon_1}(0) = \begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \\ 5 \\ 7 \\ 8 \\ 9
\end{array}
\begin{array}{cc|ccc|ccc|cc}
0 & 1 & 2 & 3 & 4 & 6 & 5 & 7 & 8 & 9 \\
\hline
* & * & * & * & * & * & * & * & * & * \\
 & * & & 0_{\mathrm{Y}} & * & & 0_{\mathrm{Y}} & * & & * \\
\hline
 & & * & * & * & & & & * & * \\
 & & & 0_{\mathrm{Z}} & * & & & & 0_{\mathrm{X}} & * \\
 & & & & * & & & & & * \\
\hline
 & & & & & * & * & * & * & * \\
 & & & & & & 0_{\mathrm{Z}} & * & 0_{\mathrm{X}} & * \\
 & & & & & & & * & & * \\
\hline
 & & & & & & & & * & * \\
 & & & & & & & & & * \\
\hline
\end{array}
$$

The blocks denoted by 0_X (respectively 0_{Y} or 0_{Z}) are the so called *adjoint blocks*, which means that one of them can be reduced to zero by a simultaneous transformation. For instance there is only one block X (respectively Y or Z) operating on both of them.

In accordance to Section 2.7 the category of block matrices BM_P has objects $B := \mu_{\varepsilon_1}(0)$ and morphisms $S := f(0)$.

Endomorphisms

Let us analyze matrices $\frac{dF_k}{dz_0}(0)$, $f_x(0)$ and $f_y(0)$ looking for an endomorphism. On the picture blow we mark entries of the matrix $\mu_{\varepsilon_1}(0)$, where it is possible to obtain zero in two or more different ways. In Section 4.2 (see formula (4.17)) it was proven that if such a place of nonzero size exists then there exists a nontrivial endomorphism.

$$\begin{array}{c|cc|ccc|ccc|cc|}
 & 0 & 1 & 2 & 3 & 4 & 6 & 5 & 7 & 8 & 9 \\ \hline
0 & & & & & & & & & & \\
1 & & & & & & & & & & \\ \hline
2 & & & & & & & & & & \\
3 & X_0 & & & & & & & & & \\
4 & * & & & & & * & & & & \\ \hline
6 & & & & & & & & & & \\
5 & X_0 & & & & & & & & & \\
7 & * & & * & & & & & & & \\ \hline
8 & * & * & & & & & & & & \\
9 & * & * & * & Y_5 & & * & Y_5 & & & \\ \hline
\end{array}$$

If such a block (i, j) has size greater than zero (i.e. if $s_i \cdot s_j \neq 0$) then there exists a nonscalar endomorphism. We call such blocks *mutually excluding* and denote by $i \cap j$. Thus we have:

- $0, 1 \cap 8, 9$;
- $0 \cap 3, 4, 7$;
- $9 \cap 2, 3, 6$;
- $4 \cap 6$ and $7 \cap 2$.

Hence, there are ten possibilities for the block structure of the matrix $\mu_{\varepsilon_1}(0)$, namely, it can consist of the following blocks:

$$\begin{array}{rl}
(0,1,2,6) & (4,7,8,9) \\
(1,2,3,5,6) & (3,4,5,7,8) \\
(1,2,3,4,5) & (3,5,6,7,8) \\
(1,3,4,5,7) & (2,3,5,6,8) \\
(1,3,5,6,7) & (2,4,3,5,8),
\end{array}$$

where we couple dual forms together. Note that $s_3 = s_5$ and in each of the listed cases the matrix problem BM_P can be written for nine blocks instead of ten “gluing” blocks 3 and 5 together. In terms of bocses and biquivers this means that we have two copies of vertex three. Substituting the category by its skeleton we obtain a quiver on four vertices.

Recovering of sizes

The occurring configuration is unambiguously determined by the rank r and and $\bar{\mathbb{d}} := (\bar{d}_1, \bar{d}_2, \bar{d}_3)$, where $\bar{d}_k := d_k \bmod r$ for $k = 1, 2, 3$. For the following four cases objects B (or, that is equivalent, the incidence matrix of P) have the form

$$\begin{array}{c|cccc|} \multicolumn{1}{c}{} & i_1 & i_2 & i_3 & i_4 \\ \cline{2-5} i_1 & * & * & * & * \\ i_2 & & * & & \\ i_3 & & & * & \\ i_4 & & & & * \\ \cline{2-5} \end{array} \quad \text{or its dual} \quad \begin{array}{c|cccc|} \multicolumn{1}{c}{} & i_1 & i_2 & i_3 & i_4 \\ \cline{2-5} i_1 & * & & & * \\ i_2 & & * & & * \\ i_3 & & & * & * \\ i_4 & & & & * \\ \cline{2-5} \end{array} \tag{5.9}$$

1. $I = \{0, 1, 2, 6\}$ if $r > \bar{d}_1 + \bar{d}_2 + \bar{d}_3$ and $s_0 = r - (\bar{d}_1 + \bar{d}_2 + \bar{d}_3)$, $s_1 = \bar{d}_3$, $s_2 = \bar{d}_2$ and $s_6 = \bar{d}_1$;

1′. $I = \{4, 7, 8, 9\}$ if $\bar{d}_1 + \bar{d}_2 + \bar{d}_3 > 2r$ and $s_4 = r - \bar{d}_1$, $s_7 = r - \bar{d}_2$, $s_8 = r - \bar{d}_3$ and $s_9 = (\bar{d}_1 + \bar{d}_2 + \bar{d}_3) - 2r$;

2. $I = \{1, 2, 3, 6\}$ if $\bar{d}_1 + \bar{d}_2 + \bar{d}_3 > r > \bar{d}_i + \bar{d}_j$ for all $i, j \in \{1, 2, 3\}$; and $s_1 = r - (\bar{d}_1 + \bar{d}_2)$, $s_2 = r - (\bar{d}_1 + \bar{d}_3)$, $s_3 = (\bar{d}_1 + \bar{d}_2 + \bar{d}_3) - r$, and $s_6 = r - (\bar{d}_2 + \bar{d}_3)$;

2′. $I = \{3, 4, 7, 8\}$ if $\bar{d}_i + \bar{d}_j > r$ for all $i, j \in \{1, 2, 3\}$ and $2r > \bar{d}_1 + \bar{d}_2 + \bar{d}_3$ and $s_3 = 2r - (\bar{d}_1 + \bar{d}_2 + \bar{d}_3)$, $s_4 = (\bar{d}_2 + \bar{d}_3) - r$, $s_7 = (\bar{d}_1 + \bar{d}_3) - r$, $s_8 = (\bar{d}_1 + \bar{d}_2) - r$.

For the other cases B has form

$$\begin{array}{c|cccc|} \multicolumn{1}{c}{} & i_1 & i_2 & i_3 & i_4 \\ \cline{2-5} i_1 & * & & * & * \\ i_2 & & * & * & * \\ i_3 & & & * & * \\ i_4 & & & & * \\ \cline{2-5} \end{array} \quad \text{or its dual} \quad \begin{array}{c|cccc|} \multicolumn{1}{c}{} & i_1 & i_2 & i_3 & i_4 \\ \cline{2-5} i_1 & * & * & * & * \\ i_2 & & * & * & * \\ i_3 & & & * & \\ i_4 & & & & * \\ \cline{2-5} \end{array} \tag{5.10}$$

3. $I = \{1, 2, 3, 4\}$ if $(\bar{d}_2 + \bar{d}_3) > r > (\bar{d}_1 + \bar{d}_2), (\bar{d}_1 + \bar{d}_3)$ and $s_3 = \bar{d}_1$, $s_1 = r - (\bar{d}_1 + \bar{d}_2)$, $s_2 = r - (\bar{d}_1 + \bar{d}_3)$ and $s_4 = (\bar{d}_2 + \bar{d}_3) - r$;

3′. $I = \{3, 4, 7, 8\}$ if $(\bar{d}_2 + \bar{d}_3) < r < (\bar{d}_1 + \bar{d}_2), (\bar{d}_2 + \bar{d}_3)$ and $s_3 = r - \bar{d}_1$, $s_6 = r - (\bar{d}_2 + \bar{d}_3)$, $s_7 = (\bar{d}_1 + \bar{d}_3) - r$ and $s_8 = (\bar{d}_1 + \bar{d}_2) - r$;

4. $I = \{1,3,4,7\}$ if $(\bar{d}_1+\bar{d}_3),(\bar{d}_2+\bar{d}_3) > r > (\bar{d}_1+\bar{d}_2)$ and $s_3 = r-\bar{d}_3$, $s_1 = r-(\bar{d}_1+\bar{d}_2)$, $s_4 = (\bar{d}_2+\bar{d}_3)-r$ and $s_7 = (\bar{d}_1+\bar{d}_3)-r$;

4′. $I = \{2,3,6,8\}$ if $(\bar{d}_1+\bar{d}_3),(\bar{d}_2+\bar{d}_3) < r < (\bar{d}_1+\bar{d}_2)$ and $s_3 = \bar{d}_3$, $s_2 = r-(\bar{d}_1+\bar{d}_3)$, $s_6 = r-(\bar{d}_2+\bar{d}_3)$ and $s_8 = (\bar{d}_1+\bar{d}_2)-r$;

5. $I = \{1,3,6,7\}$ if $(\bar{d}_1+\bar{d}_3) > r > (\bar{d}_1+\bar{d}_2),(\bar{d}_2+\bar{d}_3)$ and $s_3 = \bar{d}_2$, $s_1 = r-(\bar{d}_1+\bar{d}_2)$, $s_6 = r-(\bar{d}_2+\bar{d}_3)$ and $s_7 = (\bar{d}_1+\bar{d}_3)-r$;

5′. $I = \{2,4,3,8\}$ if $(\bar{d}_1+\bar{d}_3) < r < (\bar{d}_1+\bar{d}_2),(\bar{d}_2+\bar{d}_3)$ and $s_3 = r-\bar{d}_2$, $s_2 = r-(\bar{d}_1+\bar{d}_3)$, $s_4 = (\bar{d}_2+\bar{d}_3)-r$ and $s_8 = (\bar{d}_1+\bar{d}_2)-r$.

In Section 7.6 we solve obtained matrix problems in terms of bocses.

5.3 Algorithm for constructing canonical forms

Algorithm 5.3.1. Let $\lambda \in \Bbbk$ and $(r,d_1,d_2,d_3) \in \mathbb{N}\times\mathbb{Z}^3$ such that $g.c.d.(r,d_1+d_2+d_3)=1$.

- By the Euclidean algorithm we find integers c_k, $\bar{d}_k$ such that $d_k = c_k r + \bar{d}_k$ for $k=1,2,3$, and recover the normalization sheaf

$$\widetilde{\mathcal{F}} = \bigoplus_{k=1}^{3} \left(\mathcal{O}_k(c_k)^{r-\bar{d}_k} \oplus \mathcal{O}_k(c_k+1)^{\bar{d}_k}\right)$$

By the formulas above, recover the poset P determining the matrix problem BM_P and sizes of blocks $\mathbb{s} := (s_1,s_2,s_3,s_4)$.

- Use the matrix problem $A^\sigma(i)$ or $B^\sigma(i,j)$ and tuple (s_1,s_2,s_3,s_4) as initial state γ and the input data for the principal automaton 7.6.5 given in section 7.6.4. Choose a path $p:\gamma\to\gamma'$ on it such that $p:\mathbb{s}\mapsto\mathbb{s}'$, where $\mathbb{s}'$ is an elementary tuple i.e. for some $1\le i\le 4$ $s'_i=1$ and $s'_j=0$ for $j\ne i$.

- To obtain a canonical form of $B\in\mathsf{BM}_P(\mathbb{s})$ we input $B(\lambda)=\lambda\in\mathsf{BM}_P(\mathbb{s}')$ on the state γ' and reverse the matrix reduction procedure along the path p. In this way, step by step we recover the canonical form.

Chapter 6

Formalization of matrix problems

This chapter is devoted to the formalism of bocses developed by the Kiev school of representation theory to formalize and generalize matrix problems and differential graded categories.

A pair $\mathscr{A} = (A, V)$ consisting of an algebra A and an A-bimodule V equipped with coalgebra structure is called a *bocs.* Representations of a bocs $\mathscr{A}$ are representations of the algebra A with additional morphisms induced by elements of V. Consequently the category of representations of the bocs $\mathscr{A}$ can have fewer isomorphism classes than the category of representations of the algebra A. The formalism of bocses is indispensable. Even starting with the category of representations of an algebra after a step of matrix reduction it can degenerate into the category of representations of a bocs, which is not an algebra.

Another approach to the theory of bocses was proposed in [Ovs99] (in subsection 20.3.16). Namely, let $M_1, \ldots, M_n$ be a finite set of Λ-modules over some finite dimensional algebra Λ, and E be the extension closure of the set $\{M_1, \ldots, M_n\}$, then E is equivalent to the category of representations of some bocs $\mathscr{A} = (A, V)$, elementary representations of which correspond to $M_1, \ldots, M_n$.

Regretfully, the category E can possess some "bad" properties. It can happen that some idempotents do not split, and the dimension of a representation is not invariant for an isoclass. To avoid these peculiarities in [KR75] Roiter and Kleiner introduced normality and triangularity conditions (that time for differential graded categories). Bocses with these additional properties are called *Roiter bocses.* It should be noted that bocses corresponding to bimodule problems always possess a structure of a Roiter bocs.

We give a formal definition of a *matrix problem* as a problem of description the isomorphism classes of indecomposable representations of some Roiter bocs. Then the matrix reduction can be considered as some formal calculus on the set of bocses. In Section 6.6 we combine the general reduction with conditions on endomorphisms to obtain a version of formal reduction for bricks.

In Section 6.7 we describe a wide class of bocses which can be wild but in each vector dimension contain at most one one-parameter family of bricks. We call such bocses BT-bocses (for brick-tame) and show that the matrix reduction preserves the BT type, and at the end a BT-bocs is reduced either to the rational algebra $\Bbbk[t]$ or the field $\Bbbk$. This makes a basis for further calculations, since all occurring classification problems of simple vector bundles and torsion free sheaves are of BT-type.

In the light of bocs theory, it would be natural to ask about brick tame-wild dichotomy analogous to Drozd's Tame and Wild Theorem for indecomposable representations. However, at the moment we are not able to prove such a statement. It would be a subject for our further investigation.

6.1 Introduction to categorical language

$\Bbbk$-linear categories

The categorical language turns out to be well adapted for theoretical purposes. In this section we think of algebras as $\Bbbk$-linear categories and reformulate some classical definitions of representation theory in categorical terms.

Although most definitions are valid for any base field, we assume from the very beginning that $\Bbbk$ is algebraically closed in order to avoid complications later on.

Recall that a *preadditive category* A is a category with the following properties: every set of morphisms $A(\bar{\jmath}, \bar{\imath})$ is an abelian group and composition of morphisms is bilinear. A category is called *additive* if it is preadditive and contains direct sums of any two objects.

A category A is called $\Bbbk$-*linear* if every set of morphisms $A(\bar{\imath}, \bar{\jmath})$ is a finite dimensional $\Bbbk$-vector space and the composition of morphisms

$$A(\jmath, \imath) \times A(\kappa, \jmath) \to A(\kappa, \imath)$$

is $\Bbbk$-bilinear for all objects $\bar{\imath}, \bar{\jmath}$ and $\bar{\kappa}$. A functor $F : A \to B$ of $\Bbbk$-linear categories is called $\Bbbk$-*linear* if the induced map $F : A(\bar{\imath}, \bar{\jmath}) \to B(F(\bar{\imath}), F(\bar{\jmath}))$ is $\Bbbk$-linear.

In what follows all categories will be assumed to be $\Bbbk$-linear with finite dimensional morphism spaces (that is all functors will be $\Bbbk$-linear).

The following example reflects the correspondence between $\Bbbk$-algebras and $\Bbbk$-linear categories.

Example 6.1.1. A $\Bbbk$-algebra can be considered as a $\Bbbk$-linear category with finitely many indecomposable objects, and vice versa. Namely, let A be a $\Bbbk$-algebra, then we can regard the category A with a unique object $*$, and morphisms $a : * \to *$ are elements of the algebra A. Moreover, if $e_1 + \cdots + e_n = 1$ is a decomposition of the identity in A, then we can consider the category with objects $1, \ldots, n$ and morphisms $A(i, j) = e_j A e_i$.

Keeping with algebraic terminology, define a *left module* M over a category A to be a $\Bbbk$-linear functor $M : A \to \mathsf{Vect}_\Bbbk$, where $\mathsf{Vect}_\Bbbk$ is the category of finite dimensional $\Bbbk$-vector spaces [1]. Respectively, define a *right module* to be a $\Bbbk$-linear functor $A^\circ \to \mathsf{Vect}_\Bbbk$, where A° is the category *opposite* or *dual* to A, that is the category with the same objects as A but reverse morphisms, i.e. a right module is a contravariant functor from A to $\mathsf{Vect}_\Bbbk$. An A-B-bimodule is a bilinear functor

$$M : B^\circ \times A \to \mathsf{Vect}_\Bbbk,$$

Consequently, an A-A-bimodule is a bilinear functor

$$M : A^\circ \times A \to \mathsf{Vect}_\Bbbk .$$

The category of *left* A-modules is called the *category of representations* of A and is denoted by $\mathsf{Rep}(A)$.

Operations with categories

Let $M : B^\circ \times A \to \mathsf{Vect}_\Bbbk$, mapping $(\imath_B, \imath_A) \mapsto M(\imath_B, \imath_A)$ for all objects $\imath_A \in Ob(A), \imath_B \in Ob(B)$, be an A-B-bimodule and let $N : A^\circ \times C \to \mathsf{Vect}_\Bbbk$, mapping $(\imath_A, \imath_C) \mapsto N(\imath_A, \imath_C)$ for all objects $\imath_A \in Ob(A), \imath_C \in Ob(C)$, be a C-A-bimodule, then we define the *tensor product* $N \otimes_A M$ over A to be a C-B-bimodule as follows:

$$\big(N \otimes_A M\big)(\imath_B, \imath_C) := \Big(\bigoplus_{\imath_A \in Ob(A)} N(\imath_A, \imath_C) \otimes_\Bbbk M(\imath_B, \imath_A)\Big)/\sim,$$

where "$\sim$" is an equivalence relation given by $na \otimes m - n \otimes am$, for $n \in N(\imath'_A, \imath_C)$, $m \in M(\imath_B, \imath_A)$ and $a \in A(\imath_A, \imath'_A)$.

Let $L : A^\circ \times B \to \mathsf{Vect}_\Bbbk$ be a B-A-bimodule. Define a C-B-bimodule $\mathrm{Hom}_A(L, N)$ as

$$\Big(\mathrm{Hom}_A(L, N)\Big)(\imath_B, \imath_C) := \mathrm{Hom}_A(L(\imath_B, -), N(\imath_C, -)).$$

Let $F : A \to B$ be a functor from the category A to the category B. Then there is an induced functor $F^* : \mathsf{Rep}(B) \to \mathsf{Rep}(A)$, which sends a left B-module M to the left A-module ${}_AM$ defined as the composition $M \circ F$.

Analogously we can construct a right A-module M_A for a right B-module M. Since the category B itself has an A-A-bimodule structure it make sense to introduce special notations: let B_A denote B as a B-A-bimodule (that is left B-module and right A-module), ${}_AB$ denote B as an A-B-bimodule, and ${}_AB_A$ denote B as an A-A-bimodule. As in the case of modules over rings, we have:

$${}_AM = \mathrm{Hom}_B(B_A, M) \cong_A B \otimes_B M$$

[1] for our purposes we assume all representations to be finite dimensional, however, most definitions are valid for the infinite dimensional case also

and
$$M_A = \mathrm{Hom}_B({}_AB, M) \cong M \otimes_B B_A.$$
Let N be an A-A-bimodule; then we can consider a B-A-bimodule ${}^BN := B_A \otimes_A N$, an A-B-bimodule $N^B := N \otimes_A {}_AB$, and a B-B-bimodule
$${}^BN^B := B_A \otimes_A N \otimes_A {}_AB.$$
Note that $A^B = A \otimes_A {}_AB \cong {}_AB$, ${}^BA := B_A \otimes_A A \cong B_A$ and
$${}^BA^B := B_A \otimes_A A \otimes_A {}_AB \cong B_A \otimes_A {}_AB \xrightarrow{\nu} B,$$
where ν is a the natural projection induced by the multiplication in B.

Skeleton and additive hull.

Recall that a morphism $a \in A(\bar{\imath}, \bar{\imath})$ is called *idempotent* if $a^2 = a$. An idempotent a splits if there exist morphisms $b : \bar{\imath} \to \bar{\jmath}$ and $c : \bar{\jmath} \to \bar{\imath}$ such that $c \circ b = a$ and $b \circ c = id_{\bar{\jmath}}$. An additive category is called *Karoubian* or *fully additive* if all its idempotents split.

For an additive category A one can construct its *Karoubization* $\mathsf{add}(A) \supset A$, as follows: objects of $\mathsf{add}(A)$ are pairs $(\bar{\imath}, e_{\bar{\imath}})$, consisting of $\bar{\imath} \in Ob(A)$ and idempotents $e_{\bar{\imath}} \in A(\bar{\imath}, \bar{\imath})$, and the set of morphisms $\mathsf{add}(A)((\bar{\imath}, e_{\bar{\imath}}), (\bar{\jmath}, e_{\bar{\jmath}}))$ is $e_{\bar{\jmath}}A(i, j)e_{\bar{\imath}}$. One can easily check that any idempotent in $\mathsf{add}(A)$ splits. It turns out that the constructed category is minimal karoubian category containing A. Some times $\mathsf{add}(A)$ is also called the category of matrices over A, *additivization* or the *additive hull* of A.

In what follows for a non-additive category A we consider the additive category A' generated by A, and $\mathsf{add}(A)$ which is the additive hull of A'. Hence, $\mathsf{add}(A)$ is defined for any $\Bbbk$ category A. A linear functor $F : A \to B$ extends uniquely to the linear functor from $\mathsf{add}(A)$ to $\mathsf{add}(B)$.

An object $\bar{\imath}$ of a category is called *decomposable* if it is isomorphic to a direct sum of two other objects; if there is no such objects for $\bar{\imath}$, it is called *indecomposable.* One can prove that a finite dimensional k-linear karoubian category A is a *Krull-Schmidt* category, i.e. any object $\bar{\imath}$ has a unique finite decomposition into indecomposable ones: $\imath = \oplus i^{\alpha(i)}$.

A category A is called *skeletal* if all its objects are indecomposable in $\mathsf{add}(A)$, in other words, it contains no nontrivial idempotents, and any two objects are nonisomorphic. For any category A we construct a skeletal category $\mathsf{sk}(A)$, by taking the restriction of A on the set $I := ind(A)$, which is the set of isomorphism classes of indecomposable objects of $\mathsf{add}(A)$. This category is called the *skeleton* of A.

Two categories are called *Morita-equivalent* if their categories of representations are equivalent. In order to study representations we will often replace a category by a Morita-equivalent one. Note that categories A and B are Morita-equivalent if and only if $\mathsf{sk}(A)$ and $\mathsf{sk}(B)$ are isomorphic. Clearly, the categories A, $\mathsf{sk}(A)$ and $\mathsf{add}(A)$ are Morita-equivalent. Often speaking about modules we do not stress the difference between these categories and abusing notations write $F : A \to B$ for a functor $F : A \to \mathsf{add}(B)$.

For our further investigations we assume that the set of classes of indecomposable objects $I = \{1, \ldots, n\}$ is finite.

Example 6.1.2. If the decomposition of the identity $e_1 + \cdots + e_n = 1$ in Example 6.1.1 is minimal, then the constructed objects $1, \ldots, n$ are indecomposable, but some of them can be isomorphic. By taking one copy from each isomorphism class, and restricting the category to this set of representatives, we obtain a skeletal category. All constructed categories are Morita-equivalent, thus we can consider any of them instead of the algebra A.

On the other hand, if A is a skeletal category with objects $1, \ldots, n$, then we can consider a $\Bbbk$-algebra A with a decomposition $1_A = e_1 + \cdots + e_n$, where e_i is the identity morphism of the object $i \in I$. Elements of A are generated by morphisms $a \in A(i, j)$ and the multiplication is defined as the superposition:

$$a \cdot b = \begin{cases} ab, & \text{if the source of } a \text{ coincides with the target of } b \\ 0, & \text{otherwise.} \end{cases}$$

The skeleton of a category provides a clear graphical description of its structure. Recall that a *quiver* Q with the set of vertices $I = \{1, \ldots, n\}$, is an oriented graph, possibly with multiple arrows and loops. The set of arrows from a vertex i to a vertex j is denoted by $Q(i, j)$. A *path* p from a vertex i to a vertex j is a sequence of arrows $p = a_m \ldots a_2 a_1$, where a_t are arrows of Q such that the source of a_1 is i, the target of a_m is j and the source of a_{t+1} coincides with the target of a_t for all $t = 1, \ldots, m-1$. To the quiver Q we can assign its *path category* $\Bbbk Q$, which objects are vertices $i \in I$ and morphisms of $\Bbbk Q(i, j)$ are linear combinations of the paths from i to j. The multiplication $\Bbbk Q(i, j) \times \Bbbk Q(k, j) \to \Bbbk Q(k, j)$ is given by the concatenation of paths and $\Bbbk$-linearity.

A category A is called *free* if it is isomorphic to the category of paths $\Bbbk Q$ of some quiver Q. The images of arrows of Q under the isomorphism $\Bbbk Q \to A$ form *a set of free generators* of the category A. Two free categories are isomorphic if and only if they have isomorphic quivers, hence, there is a one-to-one correspondence between quivers and free algebras. On the other hand, the set of free generators in a free category A can be chosen in many ways, but the quivers are isomorphic.

Recall that the category of representations $\mathsf{Rep}(Q)$ of a quiver Q with vertices $1, \ldots, n$ is defined as follows:

- a representation M is given by a collection of vector spaces $M(1), \dots M(n)$ and linear maps $M(a) : M(i) \to M(j)$ for arrows $a : i \to j$;
- a morphism $S = (S_1, \dots, S_n) : M \to M'$ is defined by linear maps $S_i : M(i) \to M'(i)$ of vector spaces $M(i)$ and $M'(i)$.

From this definition obviously follows that $\mathsf{Rep}(Q)$ is isomorphic to $\mathsf{Rep}(\Bbbk Q)$.

Dimension of a representation

Here we give some definitions concerning the dimension of a representation, which will be important later for induction arguments. By the dimension of a vector space we mean its dimension over $\Bbbk$, i.e. $\dim := \dim_{\Bbbk}$. If M is a representation of a category A with finitely many indecomposable objects $I = \{1, \dots, n\}$, then its *vector dimension* is the tuple

$$\underline{\dim}(M) := \Big(\dim(M(1)), \dim(M(2)) \dots, \dim(M(n))\Big) \in \mathbb{N}^n \tag{6.1}$$

and the *dimension* of M is the sum: $\dim(M) := \sum_{i \in I} \dim(M(i))$. Obviously, $\mathsf{Rep}(A)$ can be stratified using the vector dimension:

$$\mathsf{Rep}(A) = \bigcup_{\mathbb{s}} \mathsf{Rep}(A)(\mathbb{s}), \tag{6.2}$$

where $\mathbb{s} = (s_1, \dots, s_n) \in \mathbb{N}^n$ and $\mathsf{Rep}(A)(\mathbb{s})$ denotes the full subcategory of $\mathsf{Rep}(A)$ consisting of representations of vector dimension $\mathbb{s}$.

A representation M is called *sincere at a vertex* $i \in I$ if $\dim(M(i)) > 0$, and it is called *sincere* if it is sincere for each $i \in I$.

Besides the definition of the dimension and the vector dimension an important role plays the so called norm of a representation, since it enables us to compare representations of different quivers.

Definition 6.1.3. A quiver Q determines a norm:

$$\begin{aligned} \|\cdot\| : \mathbb{N}^n &\to \mathbb{N} \\ \mathbb{s} &\mapsto \sum_{(i \to j) \in Q} s_i s_j \end{aligned} \tag{6.3}$$

The *norm of a representation* M is defined as

$$\|M\| = \sum_{(i \to j) \in Q} \dim(M(i)) \cdot \dim(M(j)).$$

Remark 6.1.4. Note that all the definitions can be translated to the language of differential biquivers and Roiter bocses, which will be considered below. In that case the norm of a representation is equal to the negative part of the Tits form. We refer to [Dro01] for details.

Let us give some examples:

Example 6.1.5. Consider the quiver Q with a unique loop t : $\bullet \overset{t}{\circlearrowleft}$ The corresponding path algebra A is isomorphic to $\Bbbk[t]$. Indecomposable finite dimensional $\Bbbk[t]$-modules are $\Bbbk[t]/(t-\lambda)^r$ for $\lambda \in \Bbbk$, $r \in \mathbb{N}$. Rewriting them as representations of the quiver Q, we get Jordan blocks $J_r(\lambda)$.

Example 6.1.6. Let Q be the quiver

$$\begin{tikzcd} & \overset{1}{\bullet} & \\ \overset{2}{\bullet} \arrow[ur, "a"] & & \overset{3}{\bullet} \arrow[ul, "b"'] \end{tikzcd}$$

then a representation $M \in \mathsf{Rep}(Q)$ is given by three vector spaces $M(1), M(2)$, and $M(3)$ and two linear maps corresponding to the arrows a and b. For simplicity we write

$$M = (M(a), M(b)).$$

Two representations M and M' are isomorphic if there exist automorphisms S_i of $M(i)$ such that $M'(a) = S_1^{-1}M(a)S_2$ and $M'(b) = S_1^{-1}M(b)S_3$. In other words, the matrix M' can be obtained from M by simultaneous row transformation and independent column transformations for each block $M(a)$ and $M(b)$:

$$\begin{aligned} M(a) &\mapsto S_1^{-1}M(a)S_2, \\ M(b) &\mapsto S_1^{-1}M(b)S_3. \end{aligned} \tag{6.4}$$

6.2 Bocses

A pair $\mathscr{A} := (A, V)$ is called a *bocs* provided that A is a category and V is a *coalgebra* over A. That is: V is an A-A-bimodule together with two A-bimodule homomorphisms: *counit* $\varepsilon : V \to A$, and *comultiplication* $\mu : V \to V \otimes_A V$, satisfying coassociativity and counitary laws:

$$(id_V \otimes \mu) \circ \mu = (\mu \otimes id_V) \circ \mu \quad \text{and} \quad (id_V \otimes \varepsilon) \circ \mu = (\varepsilon \otimes id_V) \circ \mu = id_V :$$

$$\begin{tikzcd} V \arrow[r, "\mu"] \arrow[d, "\mu"'] & V \otimes_A V \arrow[d, "\mu\otimes id_V"] \\ V \otimes_A V \arrow[r, "id_V\otimes\mu"] & V \otimes_A V \otimes_A V \end{tikzcd} \qquad \begin{tikzcd} & A \otimes_A V \\ V \arrow[ur, bend left, "iso"] \arrow[r, "\mu"] \arrow[dr, bend right, "iso"'] & V \otimes_A V \arrow[u, "\varepsilon\otimes id_V"'] \arrow[d, "id_V\otimes\varepsilon"] \\ & V \otimes_A A \end{tikzcd}$$

To a bocs $\mathscr{A}$ we can assign its *category of representations* $\mathsf{Rep}(\mathscr{A})$, defined as follows:

- objects are representations of A,
- morphisms are defined as

$$\mathrm{Hom}_{\mathscr{A}}(M, N) := \mathrm{Hom}_A(V \otimes_A M, N).$$

A composition of two morphisms $\psi : V \otimes_A N \to L$ and $\varphi : V \otimes_A M \to N$ is defined as

$$V \otimes_A M \xrightarrow{\mu \otimes id_M} V \otimes_A V \otimes_A M \xrightarrow{id \otimes \varphi} V \otimes_A N \xrightarrow{\psi} L.$$

If $V = A$ and the counit and the comultiplication are the identity and diagonal maps respectively, then the bocs $\mathscr{A} = (A, V)$ is called *principal* and is denoted by A, since $\mathsf{Rep}(\mathscr{A}) = \mathsf{Rep}(A)$.

Remark 6.2.1. The category of representations of a bocs $\mathscr{A} = (A, V)$ modifies the original category of representations $\mathsf{Rep}(A)$. However, in contrast to $\mathsf{Rep}(A)$ the category $\mathsf{Rep}(\mathscr{A})$ is not abelian and even not always fully additive, though it is obviously additive.

A *morphism* $F := (F_0, F_1)$ of two bocses $\mathscr{A} = (A, V)$ and $\mathscr{B} = (B, W)$ consists of a functor $F_0 : A \to B$ and an A-homomorphism $F_1 : V \to_A W_A$ such that

$$F_0 \circ \varepsilon_{\mathscr{A}} = \varepsilon_{\mathscr{B}} \circ F_1 \quad \text{and} \quad \tilde{\nu} \circ F_1 \otimes F_1 \circ \mu_{\mathscr{A}} = \mu_{\mathscr{B}} \circ F_1 :$$

$$\begin{tikzcd} V \arrow[r, "\varepsilon_{\mathscr{A}}"] \arrow[d, "F_1"'] & A \arrow[d, "F_0"] \\ {}_AW_A \arrow[r, "\varepsilon_{\mathscr{B}}"] & B \end{tikzcd} \qquad \begin{tikzcd} & V \otimes_A V \arrow[r, "F_1 \otimes F_1"] & {}_AW_A \otimes_A {}_AW_A \arrow[dr, "\tilde{\nu}"] & \\ V \arrow[ur, "\mu_{\mathscr{A}}"] \arrow[r, "F_1"] & {}_AW_A \arrow[rr, "\mu_{\mathscr{B}}"] & & W \otimes_B W \end{tikzcd}$$

where $\tilde{\nu} : W_A \otimes_A {}_AW \to W \otimes_B W$ is the map induced by the natural projection $\nu : B_A \otimes_A {}_AB \to B$.

A morphism of bocses $(F_0, F_1) : \mathscr{A} \to \mathscr{B}$ induces a functor

$$F^* : \mathsf{Rep}(\mathscr{B}) \to \mathsf{Rep}(\mathscr{A}),$$

sending a representation M into ${}_AM$ and a morphism $\alpha : W \otimes_B M \to N$ into the composition

$$V \otimes_A {}_AM \xrightarrow{F_1 \otimes id} ({}_AW_A) \otimes_A ({}_AM) \xrightarrow{\tilde{\nu}}{}_A\, W \otimes_B M \xrightarrow{\alpha}{}_A\, N, \tag{6.5}$$

where, as above $\tilde{\nu} : W_A \otimes_A {}_AM \to W \otimes_B M$ is the map induced by the natural projection $\nu : B_A \otimes_A {}_AB \to B$.

Remark 6.2.2. In general F^* does not preserve isomorphism classes of objects. Indeed, if $M \cong N$ and $\alpha : W \otimes_B M \to N$ is an isomorphism, then $F^*(\alpha)$ is not an isomorphism provided that the composition $\tilde{\nu} \circ (F_1 \otimes id)$ is not an isomorphism. Nevertheless, the functor F^* is well behaved in the situations considered below.

Base change for representations of bocses

If $\mathscr{A} = (A, V)$ is a bocs and $F : A \to B$ a functor then we can define the *induced* bocs $\mathscr{A}^B := (B, {}^B V^B)$, with counit

$$\varepsilon_{\mathscr{A}^B} : \ {}^B V^B \xrightarrow{\varepsilon} {}^B A^B \xrightarrow{\nu} B \tag{6.6}$$

and comultiplication

$$\begin{aligned}\mu_{\mathscr{A}^B} : {}^B V^B &\xrightarrow{\quad\mu\quad} {}^B V \otimes_A V^B \cong {}^B V \otimes_A A \otimes_A V^B \\ &\xrightarrow{id_V \otimes F_0 \otimes id_V} {}^B V \otimes_A {}_A B_A \otimes_A V^B \cong {}^B V^B \otimes_B {}^B V^B.\end{aligned} \tag{6.7}$$

We denote the induced morphism of bocses also by $F := (F, F_1) : \mathscr{A} \to \mathscr{A}^B$, where F_1 is given by the composition

$$V \xrightarrow{\simeq} A \otimes_A V \otimes_A A \xrightarrow{F \otimes id \otimes F} {}_A\,({}^B V^B)_A.$$

Let us translate some definitions from algebras to bocses:

Definition 6.2.3. Let $\mathscr{A} = (A, V)$ be a bocs.

- A bocs $\mathscr{A}' := (A', V')$ is called a *subbocs* of $\mathscr{A}$ if $F_0 : A' \hookrightarrow A$ is a subcategory of A and $F_1 : V' \hookrightarrow_{A'} V_{A'}$ is an embedding of A'-modules such that (F_0, F_1) is a morphism of bocses i.e.
  $F_0 \circ \varepsilon_{\mathscr{A}'} = \varepsilon_{\mathscr{A}} \circ F_1$, and $\tilde{\nu} \circ (F_1 \otimes F_1) \circ \mu_{\mathscr{A}'} = \mu_{\mathscr{A}} \circ F_1$, where $\tilde{\nu} : V_{A'} \otimes_{A'} {}_{A'} V \to V \otimes_A V$ is the canonical embedding.
- if $F : A \to \bar{A}$ is a factor category,[2] define $\mathscr{A}^{\bar{A}}$ to be a *factor bocs*;
- the *additive hull of a bocs* is defined to be $\mathsf{add}(\mathscr{A}) := \mathscr{A}^{\mathsf{add}(A)}$.

Lemma 6.2.4. *If $\mathscr{A} = (A, V)$ is a bocs and $F : A \to B$ an arbitrary $\Bbbk$-linear functor, then the induced functor $F^* : \mathsf{Rep}(\mathscr{A}^B) \to \mathsf{Rep}(\mathscr{A})$ is fully faithful.*

[2]Here the algebraic terminology seems to be more appropriate, by a factor category we mean nothing more than a factor algebra.

Proof. To show that F is a full functor take a morphism

$$\alpha \in \operatorname{Hom}_{\mathscr{A}^B}(M,N) = \operatorname{Hom}_B({}^BV^B \otimes_B M, N),$$

then according to equation (6.5) $F^*(\alpha)$ is the composition

$$F^*(\alpha) : V \otimes_A {}_AM \xrightarrow{F_1 \otimes id} ({}_A({}^BV^B)_A) \otimes_A ({}_AM) \xrightarrow{\tilde{\nu}} {}_A({}^BV^B) \otimes_B M \xrightarrow{\alpha} {}_AN,$$

where $\tilde{\nu}$ is the canonical morphism induced by ν. Take into account that the composition of $id \otimes F$ and ν gives the identity functor

$$id : B_A \cong B_A \otimes_A A \stackrel{id\otimes F}{\longrightarrow} B_A \otimes_A {}_AB_A \stackrel{\nu}{\longrightarrow} B_A,$$

or $id : {}_AB \to {}_AB$. Hence, for a morphism $\beta \in \operatorname{Hom}_{\mathscr{A}}({}_AM, {}_AN)$ $\beta : V \otimes_A {}_AM \to_A N$ we can construct a uniquely defined morphism $\alpha \in \operatorname{Hom}_{\mathscr{A}^B}(M,N)$ taking

$$\alpha : {}^BV^B \otimes_B M \cong {}^BV \otimes_A ({}_AM) \stackrel{id\otimes\beta}{\longrightarrow} {}^B({}_AN) \stackrel{\nu\otimes id}{\longrightarrow} N,$$

such that $\beta = F^*(\alpha)$. □

For a morphisms $F : A \to B$ such that the induced functor F^* is dense, this lemma allows us to replace the bocs $\mathscr{A}$ by the Morita-equivalent bocs $\mathscr{A}^B$. To obtain a functor $F : A \to B$ with the dense induced functor F^*, we consider *push-outs,* and obtain the density condition from the universal property:

Proposition 6.2.5. *Let $\mathscr{A} = (A,V)$ be a bocs and $\mathscr{A}' = (A',V')$ its subbocs with the embedding map $i := (i_A, i_V) : \mathscr{A}' \to \mathscr{A}$. Let $B' = (B',B')$ be a principal bocs, and*

$$F' = (F', F_1') : \mathscr{A}' \to B',$$

be a morphism of bocses such that the induced functor

$$F'^* : \mathsf{Rep}(B') \to \mathsf{Rep}(\mathscr{A}'),$$

is dense. Let $F : A \to B$ be the push-out of the diagram:

$$\begin{tikzcd} A' \arrow[r, "i_A"] \arrow[d, "F'"] & A \arrow[d, "F"] \\ B' \arrow[r, "i_B"] & B. \end{tikzcd}$$

Then the functor $F^* : \mathsf{Rep}(\mathscr{A}^B) \to \mathsf{Rep}(\mathscr{A})$ is dense.

Proof. A representation $M : A \to \mathsf{Vect}_{\Bbbk}$ can be restricted to the subcategory A' of A. Since the functor F'^{*} is dense, there exists a representation $N : B' \to \mathsf{Vect}_{\Bbbk}$ such that $F'^{*}(M') = M|_{A'}$. By the universal property of the push-out there exists a unique representation $\widetilde{M} : B \to \mathsf{Vect}_{\Bbbk}$ such that $\widetilde{M} \circ F = M$ and $\widetilde{M} \circ i_B = N$.

$$\begin{tikzcd}
A' \arrow[r, "i_A"] \arrow[d, "F'"] & A \arrow[d, "F"] \arrow[ddr, bend left, "M"] & \\
B' \arrow[r, "i_B"] \arrow[ddrr, bend right, "N"'] & B \arrow[dr, "\exists ! \widetilde{M}"] & \\
 & & \mathsf{Vect}_{\Bbbk}
\end{tikzcd}$$

To complete the proof note that the category $\mathsf{Rep}(\mathscr{A})$ has the same objects as $\mathsf{Rep}(A)$, and the category $\mathsf{Rep}(\mathscr{A}^B)$ has the same objects as $\mathsf{Rep}(B)$, thus for any representation of the bocs $\mathscr{A}$ we can find the corresponding representation of the bocs $\mathscr{A}^B$. $\square$

In Subsection 6.5 we use the base change for the case when the subcategory A' has a unique generator a, and B' is the discrete category $ind(\mathsf{Rep}\,\mathscr{A}')$: it contains only trivial morphisms and its objects are the isomorphism classes of indecomposable objects of $\mathsf{Rep}\,\mathscr{A}'$.

6.3 Roiter bocses

In this section we discuss different types of bocses and finally give a formal definition of a matrix problem, viz. as a problem of describing iso-classes of representations of some special bocs. Such special bocses are called Roiter bocses. They were introduced by Roiter [Ro79] in attempt to give another formal approach to differential graded categories. In particular, as we will see in Appendix C any bimodule problem can be formulated in terms of Roiter bocses.

Section of a bocs

A *section* of a bocs $\mathscr{A} = (A, V)$ is a set

$$\mathfrak{w} = \{w_i \in V(i,i) \mid \varepsilon(w_i) = e_i, i \in I\}.$$

A section $\mathfrak{w}$ defines a left A-module homomorphism (a left section) $w_L : A \to V$, $a \mapsto aw_i$, for a solid arrow $a : i \to j$, and a right A-module homomorphism (a right section): $w_R : A \to V$, $a \mapsto w_j a$. We omit indices L and R, since each time it is clear, which map we have in mind. Define the *kernel* $\overline{V}$ of a bocs to be the kernel of the counit morphisms ε, then we obtain the decompositions:

$$V \cong wA \oplus \overline{V} \cong Aw \oplus \overline{V}. \tag{6.8}$$

Note that the image of the comultiplication μ belongs to the tensor product, which can be decomposed according to these splitting as follows:

$$\big(wA\oplus\overline{V}\big)\otimes_A\big(Aw\oplus\overline{V}\big) = \big(wA\otimes_A Aw\big)\oplus\big(wA\otimes_A\overline{V}\big)\oplus\big(\overline{V}\otimes_A Aw\big)\oplus\big(\overline{V}\otimes_A\overline{V}\big).$$

Note that:

$$\begin{aligned}\ker(1\otimes\varepsilon) &= \big(wA\otimes_A\overline{V}\big)\oplus\big(\overline{V}\otimes_A\overline{V}\big)\\ \ker(\varepsilon\otimes 1) &= \big(\overline{V}\otimes_A Aw\big)\oplus\big(\overline{V}\otimes_A\overline{V}\big).\end{aligned}$$

From the counitary law it is easy to see that for any element $v \in \overline{V}(i,j)$ and $w_i, w_j \in \mathfrak{w}$ the difference $\xi(v) := \mu(v) - w_j \otimes v - v \otimes w_i$, $\xi(w_i) := \mu(w_i) - w_i \otimes w_i$ belongs to both kernels $\ker(1\otimes\varepsilon)$ and $\ker(\varepsilon\otimes 1)$. Hence

$$\mu(v) = w_j \otimes v + v \otimes w_i + \xi(v), \tag{6.9}$$

$$\mu(w_i) = w_i \otimes w_i + \xi(w_i), \tag{6.10}$$

for some $\xi(v), \xi(w_i) \in \overline{V}\otimes\overline{V}$. A section $\mathfrak{w}$ is called *normal* or *grouplike* if $\mu(w_i) = w_i \otimes w_i$ for all $i \in I$. A bocs with a normal section is called *normal* too.

Tensor category

Let A be a category and U be an A-bimodule, then U determines the *tensor category*:

$$A[U] = \bigoplus_{n=0}^{\infty} U^{\otimes n},$$

where $U^0 := A$, $U^1 := U$ and $U^{\otimes n} := U \otimes_A \cdots \otimes_A U$ n-times, and

$$U^{\otimes n}(\imath,\jmath) := \bigoplus_{\kappa_1,\ldots,\kappa_{n-1}} U(\kappa_{n-1},\jmath)\otimes_A\cdots\otimes_A U(\kappa_1,\kappa_2)\otimes_A U(\imath,\kappa_1)/\sim,$$

where $\sim$ is the induced equivalence relation. As usual, if $u \in U^{\otimes n}$ we say that u is an element of *degree* n.

Define the *tensor category of a bocs* $\mathscr{A} = (A, V)$ to be the tensor category $A[\overline{V}]$, where $\overline{V}$ is the kernel of $\mathscr{A}$.

Biquivers

Let $Q = (Q_0, Q_1)$ be a *biquiver* on the finite set of vertices $I = \{1, \ldots, n\}$ that is a quiver with two types of arrows: solid and dotted. The set of solid arrows is denoted by Q_0 and the set of dotted arrows is denoted by Q_1. We assign degree zero to a solid arrow, degree one to a dotted arrow and extend it to paths by additivity. (I.e. the *degree* of a path is the number of dotted arrows it contains.) Denoting by $A := \Bbbk Q_0$ the path algebra and $U := \langle Q_1 \rangle_A$ the A-A-bimodule generated by paths of degree one, we obtain the tensor category

$$\Bbbk Q := A[U] = \bigoplus_{n=0}^{\infty} U^{\otimes n}$$

called the *tensor path category,* where $U^{\otimes n}$ is the A-A-bimodule generated by paths of degree n. Clearly, two tensor path categories are isomorphic if and only if their biquivers are isomorphic.

A normal bocs $\mathscr{A} = (A, V)$ is called *free* if there exists a biquiver Q such that $A[\overline{V}]$ is isomorphic (not just equivalent) to $\Bbbk Q$. The images of arrows under an isomorphism $\Bbbk Q \to A[\overline{V}]$ are called *free generators of* $\mathscr{A}$.

Differential biquivers

Let $Q = (Q_0, Q_1)$ be a biquiver. A $\Bbbk$-linear map $\partial : \Bbbk Q \to \Bbbk Q$ is called *differential* if it fulfills requirements (D1)-(D3):

(D1) ∂ raises degree by one and $\partial^2 = 0$;

(D2) $\partial(e) = 0$ if e is a trivial path;

(D3) Leibniz rule: for any two paths $x, y \in \Bbbk Q$, it holds:

$$\partial(x \cdot y) = \partial(x) \cdot y + (-1)^{\deg x} x \cdot \partial(y).$$

(It seems convenient to have several notations for multiplications of paths: $xy := x \cdot y := x \otimes y$.)

If ∂ is a differential, then the pair (Q, ∂) is called a *differential biquiver.* Note that the this notion corresponds to the notion of *free differential graded category* introduced in [KR75]. Note the following lemma:

Lemma 6.3.1. *If the Leibniz rule holds for all arrows $x, y \in Q_0 \cup Q_1$, then it holds also for all paths $x, y \in \Bbbk Q$.*

Representations of a differential biquiver

We introduce the category of representations of a biquiver $\mathsf{Rep}(Q, \partial)$ originally introduced by Roiter and Kleiner for differential graded categories following Crawley-Boevey [CB90]:

- its objects are representations of the quiver Q_0;
- morphisms are defined as follows. Let Γ be a quiver consisting of two copies of $Q_0 : Q_0$ itself and Q_0', arrows $w_i : i \to i'$, and arrows $v : i \to j'$, defined by dotted arrows $v : i \dashrightarrow j$ from Q_1. A morphism $S : M \to N$ between two representations M and N of the biquiver Q is a representation of the quiver Γ, such that its restriction on Q_0 and Q_0' are M and N respectively, and for any solid arrow $a : i \to j$ the following relation is satisfied:

$$S(\partial(a)) \;=\; N(a')S(w_i) - S(w_j)M(a). \tag{6.11}$$

Let us define a composition $T \circ S$ of two morphisms $S : M \to N$ and $T : N \to L$. For $w_i \in \mathfrak{w}$ we have

$$\big(T \circ S\big)(w_i) = T(w_i) \cdot S(w_i).$$

For a dotted arrow $v : j \dashrightarrow i$ with the differential

$$\partial(v) = \sum \alpha p_1\, u\, p_2\, u'\, p_3,$$

where p_i are solid paths, u, u' are dotted arrows such that $p_1\, u\, p_2\, u'\, p_3$ is a path from i to k and $\alpha \in \Bbbk$, define

$$\begin{aligned}\big(T \circ S\big)(v) = T(w_i) \cdot S(v) + T(v) \cdot S(w_j) \\ + \sum \alpha L(p_1) \cdot T(u) \cdot N(p_2) \cdot S(u') \cdot M(p_3).\end{aligned} \tag{6.12}$$

Remark 6.3.2. Note that if the differential of v is $\partial(v) = \sum\limits_{i \dashrightarrow k \dashrightarrow j} \alpha u \otimes u'$, then

$$\big(T \circ S\big)(v) = T(w_i) \cdot S(v) + T(v) \cdot S(w_j) + \sum_{i \dashrightarrow k \dashrightarrow j} \alpha T(u) \cdot S(u').$$

Remark 6.3.3. If (Q, d) is a differential biquiver with no dotted arrows, i.e. $Q_1 = \emptyset$, then $\mathsf{Rep}(Q, \partial) = \mathsf{Rep}(Q_0)$. Thus, any quiver can be considered as a differential biquiver.

Example 6.3.4. Let Q be the following biquiver

$$\begin{tikzcd} & 1 & \\ 2 \arrow[ur, "a"] \arrow[rr, dotted, "v"] & & 3 \arrow[ul, "b"'] \end{tikzcd}$$

with the differential $\partial(b) = \partial(v) = 0$ and $\partial(a) = b \cdot v$. Then a morphism of representations $S : M \to N$ is a representation of a biquiver Γ :

$$\begin{tikzcd} 2 \arrow[r, dotted, "w_2"] \arrow[d, "a"'] \arrow[ddr, dotted, "v"] & 2' \arrow[d, "a'"] \\ 1 \arrow[r, dotted, "w_1"] & 1' \\ 3 \arrow[u, "b"] \arrow[r, dotted, "w_3"] & 3' \arrow[u, "b'"'] \end{tikzcd}$$

such that $S(a) = M(a)$, $S(b) = M(b)$, $S(a') = N(a')$, $S(b') = N(b')$ and for $S_i := S(w_i)$ we have:

$$\begin{aligned} 0 &= N(b')S_3 - S_1 M(b), \\ N(b')S(v) &= N(a')S_2 - S_1 M(a). \end{aligned}$$

Correspondence between normal free bocses and differential biquivers

Theorem 6.3.5 ([Ro79]). *There is a one-to-one correspondence between free normal bocses and differential biquivers.*

Proof. Let $\mathscr{A} = (A, V)$ be a free bocs with a normal section $\mathfrak{w}$. Construct a differential biquiver (Q, ∂) as follows. Since $\mathscr{A}$ is free, there exists a biquiver $Q = (Q_0, Q_1)$ such that $A[\overline{V}] \cong \Bbbk Q$. A normal section $\mathfrak{w}$ of a bocs $\mathscr{A}$ defines two $\Bbbk$-linear maps:

- the map $\partial_0 : A \to \overline{V}$ taking an element $a \in A(i,j)$ to

$$\partial_0(a) = aw_i - w_j a, \tag{6.13}$$

  (the image of ∂_0 is contained in the kernel of the bocs $\overline{V}$, since $\varepsilon(aw_i - w_j a) = ae_i - e_j a = 0$)

- and $\partial_1 : \overline{V} \to \overline{V} \otimes_A \overline{V}$, taking an element $v \in \overline{V}(i,j)$ to

$$\partial_1(v) = \mu(v) - v \otimes w_i - w_j \otimes v. \tag{6.14}$$

  (The formulae (6.9) implies that the morphism ∂_1 is well defined.)

Note that the Leibnitz rule (D3) is satisfied for the maps ∂_0 and ∂_1, indeed for $a \in A(i,j)$ $b \in A(j,k)$ and $v \in \overline{V}(j,k)$ $u \in \overline{V}(i,j)$ we have:

$$\begin{aligned}
\partial_0(ba) &= baw_i - w_k ba = baw_i - bw_j a + bw_j a - w_k ba \\
&= b\partial_0(a) + \partial_0(b)a, \\
\partial_1(va) &= \mu(va) - va \otimes w_i - w_k \otimes va \\
&= \mu(v)a - v \otimes aw_i + v \otimes w_j a - v \otimes w_j a - w_k \otimes va \\
&= \partial_1(v)a - v\partial_0(a), \\
\partial_1(bu) &= \mu(bu) - bu \otimes w_i - w_k \otimes bu \\
&= b\mu(u) - bu \otimes w_i - bw_j \otimes u + bw_j \otimes u - w_k b \otimes u \\
&= b\partial_1(u) + \partial_0(b)a.
\end{aligned}$$

By the Leibnitz rule the morphisms ∂_0 and ∂_1 can be extended to a $\Bbbk$-linear map $\partial : A[\overline{V}] \to A[\overline{V}]$, such that $\partial|_A = \partial_0$, $\partial|_{\overline{V}} = \partial_1$. Since the property (D2) is automatically satisfied, thus in order to claim that ∂ is a differential it is enough to check that $\partial^2 = 0$. We verify the property $\partial^2 = 0$ for elements $a \in A$ and $v \in \overline{V}$. Since $deg(\partial(x)) = deg(x) + 1$, thus by Leibniz rule if $\partial^2(x) = \partial^2(y) = 0$, then

$$\partial^2(xy) = \partial^2(x) + (-1)^{deg(\partial(x))}\partial(x) \otimes \partial(y) + (-1)^{deg(x)}\partial(x) \otimes \partial(y) + \partial^2(y) = 0.$$

For $a \in A(i,j)$ the property follows immediately from the normality condition:

$$\begin{aligned}
\partial^2(a) &= \partial_1(aw_i - w_j a) = \mu(aw_i - w_j a) - (aw_i - w_j a) \otimes w_i - w_j \otimes (aw_i - w_j a) \\
&= aw_i \otimes w_i - w_j \otimes w_j a - aw_i \otimes w_i + w_j a \otimes w_i - w_j \otimes aw_i + w_j \otimes w_j a \\
&= 0
\end{aligned}$$

For $v \in \overline{V}(i,j)$ assume $\partial(v) = \sum\limits_k u_k \otimes u'_k$, where $u_k \in \overline{V}(k,j)$ and $u'_k \in \overline{V}(i,k)$, then:

$$
\begin{aligned}
\partial^2(v) =& \partial\big(\textstyle\sum_k u_k \otimes u'_k\big) \\
=& \sum_k \big(\partial(u_k) \otimes u'_k - u_k \otimes \partial(u'_k)\big) && \text{(change } \partial(u_k) \text{ to } \mu(u_k)) \\
=& \sum_k \Big((\mu(u_k) - w_j \otimes u_k - u_k \otimes w_k) \otimes u'_k \\
& - u_k \otimes (\mu(u'_k) - w_k \otimes u'_k - u'_k \otimes w_i)\Big) \\
=& \sum_k \Big(\mu(u_k) \otimes u'_k - u_k \otimes \mu(u'_k)\Big) \\
& + \partial(v) \otimes w_i - w_j \otimes \partial(v) && \text{(change } \partial(v) \text{ to } \mu(v)) \\
=& \sum_k \Big(\mu(u_k) \otimes u'_k - u_k \otimes \mu(u'_k)\Big) \\
& + \mu(v) \otimes w_i - w_j \otimes \mu(v) && \text{(reordering summands)} \\
& + w_j \otimes w_j \otimes v - v \otimes w_i \otimes w_i \\
=& \textstyle\sum_k \mu(u_k) \otimes u'_k + \mu(v) \otimes w_i + \mu(w_j) \otimes v && (= \mu \otimes id_V \circ \mu(v)) \\
& - \Big(\textstyle\sum_k u_k \otimes \mu(u'_k) + w_j \otimes \mu(v) + v \otimes \mu(w_i)\Big) && (= \mathrm{id}_V \otimes \mu \circ \mu(v)) \\
=& \big((\mu \otimes id_V - id_V \otimes \mu) \circ \mu\big)(v) \\
=& 0.
\end{aligned}
$$

Thus, the coassociativity law implies the property $\partial^2(v) = 0$. In this way we construct a differential biquiver from a normal free bocs.

On the other hand, assume that a differential biquiver (Q, ∂) is given. Our aim is to construct the corresponding normal free bocs. Let $A := \Bbbk Q_0$ be a category generated by Q_0 and $\overline{V} = \langle Q_1 \rangle_A$ be a free A-A-bimodule generated by Q_1. Define the A-A-bimodule $V := A \oplus \overline{V}$ with canonical embedding $w_L : A \hookrightarrow V$ and another embedding $w_R : A \hookrightarrow V$ given by the rule $w_R(a) = w_L(a) + \partial(a)$. Embeddings w_L and w_R define left A-module and right A-module structures of V respectively. Indeed, for $(a, v) \in A \oplus \overline{V}$ and $a', a'' \in A$: $a *_L (a, v) = (a'a, a'v)$

and $(a, v) *_R a' = (aa', va' - a\partial(a'))$ together with associativity properties

$$\begin{aligned}((a, v) *_R a') *_R a'' &= (aa'a'', va'a'' - a\partial(a')a'' - aa'\partial(a'')) = (a, v) *_R (a'a'')\\ \big(a' *_L (a, v)\big) *_R a'' &= (a'aa'', a'va'' - a'a\partial(a'')) = a' *_L \big((a, v) *_R a''\big).\end{aligned}$$

Hence, V is an A-A-bimodule. For each trivial path e_i consider its image $w_i := w_L(e_i) = w_R(e_i)$ in V. To endow V with an A-coalgebra structure we define the counit $\varepsilon : V \to A$ by taking $w_i \mapsto e_i$ and $\ker(\varepsilon) := \overline{V}$; (in other words $\varepsilon(w_L) = \varepsilon(w_R) = id_A$); and define the comultiplication $\mu : V \to V \otimes V$ by $\mu(w_i) := w_i \otimes w_i$ and $\mu(v) := \partial(v) + w_j \otimes v + v \otimes w_i$ for $v \in \overline{V}(i, j)$. Obviously, the counitary low holds: $(1 \otimes \varepsilon) \circ \mu(v) = (1 \otimes \varepsilon) \circ (\partial(v) + w_j \otimes v + v \otimes w_i) = v$, for $v \in (\overline{V})$, $(1 \otimes \varepsilon) \circ \mu(w_i) = (1 \otimes \varepsilon)(w_i \otimes w_i) = w_i$ and analogously, $(\varepsilon \otimes 1) \circ \mu = id_V$. The coassociativity low can be derived from the property $\partial^2 = 0$ reversing the calculations above. Hence, we have constructed a normal free bocs $\mathscr{A} = (A, V)$. □

Analogously to the case of algebras, one can show that the categories $\mathsf{Rep}(Q, \partial)$ and $\mathsf{Rep}(\mathscr{A})$, where $\mathscr{A}$ is the normal free bocs corresponding to (Q, ∂), are equivalent. Therefore, we have two parallel descriptions of the same notion: normal free bocses and differential biquivers. The language of bocses is convenient for theoretical purposes, at the same time the language of biquivers, due to its visual clearness, is well adapted for calculations. Thus, formulating a concrete problem, we speak almost only about differential biquivers.

Roiter bocses

However, even normal free bocses can have some nasty properties such as nonsplitting idempotents. Let us illustrate it by an example:

Example 6.3.6. Let (Q, ∂) be the following biquiver

$$\begin{tikzcd} \overset{2}{\bullet} \arrow[r, bend right, "a"'] & \overset{1}{\bullet} \arrow[l, dotted, bend right, "v"'] \end{tikzcd}$$

with the differential $\partial(a) = 0$ and $\partial(v) = vav$. The differential is correctly defined, since $\partial^2(v) = \partial(vav) = \partial(v)av - v\partial(av) = vavav - 0 - vavav = 0$. Indecomposable representations of (Q, ∂) are $L_1 := 0 \to \Bbbk$ and $L_2 := \Bbbk \to 0$, and $M := \Bbbk \xrightarrow{\text{id}} \Bbbk$. Indeed, assume $M = L_1 \oplus L_2$, then there should exist a nontrivial morphism $S : M \to L_1$ such that $S_1 = \text{id}$, $S_2 = 0$, and since $\partial(a) = 0$ the following diagram

$$\begin{tikzcd} \Bbbk \arrow[r, dotted, "\text{id}"] \arrow[dr, dotted, "0"] & \Bbbk \\ \Bbbk \arrow[u, "\text{id}"] \arrow[r, dotted, "0"] & 0 \arrow[u, "0"'] \end{tikzcd}$$

should be commutative, what is obviously not the case. Hence, M is an indecomposable representation. Consider the endomorphism $E : M \to M$ given by $E_1 = E_2 = 0$ and $E(v) := \mathrm{id}_{\mathbb{k}}$. It is an idempotent, as the composition $E^2 := E \circ E$:

$$\begin{tikzcd}
\mathbb{k} \arrow[r, dotted, "0"] \arrow[dr, dotted, "\mathrm{id}"] & \mathbb{k} \arrow[r, dotted, "0"] \arrow[dr, dotted, "\mathrm{id}"] & \mathbb{k} \\
\mathbb{k} \arrow[u, "\mathrm{id}"] \arrow[r, dotted, "0"] & \mathbb{k} \arrow[u, "\mathrm{id}"'] \arrow[r, dotted, "0"] & \mathbb{k} \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

consists of $E_1^2 = E_2^2 = 0$ and $E^2(v) = E_2E(v) + E(v)E_1 + E(v)M(a)E(v) = \mathrm{id}_{\mathbb{k}}\,\mathrm{id}_{\mathbb{k}}\,\mathrm{id}_{\mathbb{k}} = \mathrm{id}_{\mathbb{k}}$. Since M is indecomposable E does not split. Hence, $\mathsf{Rep}(Q, \partial)$ is not karoubian.

To avoid these peculiarities Roiter and Kleiner introduced in [KR75] an additional property:

(D4)* triangularity: there is a *level map* $h : Q \to \mathbb{N}$ such that the differential of any arrow involves only arrows of strictly smaller level.

Another important property is the *linearity* condition.

(D5)* linearity: the differential of any arrow is a linear combination of paths of lengths at most 2,

Obviously, the triangularity and linearity properties can be formulated in terms of generators of a normal free bocs. Bocses with these properties are called *triangular* and *linear* respectively.

Remark 6.3.7. The linearity property simplifies the differential. Normal free triangular and linear bocses generalize the concept of *bimodule problems*. In Appendix C we give a formal definition of a bimodule problem and prove the corresponding statement. Unfortunately, in the course of matrix reduction the linearity property can be affected.

Let us consider properties of triangular differential biquivers.

Lemma 6.3.8. *Let (Q, ∂) be a triangular differential biquiver and $S : M \to N$ be a morphism of representations $M, N \in \mathsf{Rep}(Q, \partial)$. Then S is an isomorphism if and only if the maps $S_1, \dots S_n$ are isomorphisms.*

Proof. In one direction the lemma is obvious. Assume S is a morphism with invertible $S_1, \dots S_n$. Construct the morphism S' as follows: for a vertex $i \in I$, take $S'_i := (S_i)^{-1}$, for a minimal arrow u take $S'(u) := -S_j^{-1}S(u)S'_i$ and for an arbitrary arrow v with the differential $\partial(v) = \sum \alpha p_1 u p_2 u' p_3$ define $S'(v)$

by induction. That is assume that $S'(u')$ is already defined for each u' with $h(u') < h(u)$ and take

$$S'(v) := S_j^{-1}\big(- S(v)S_i' - \sum \alpha M(p_1)S(u)N(p_2)S'(u')M(p_3)\big).$$

Applying (6.12) we see that the composition $S \circ S'$ is the identity morphism of M. Analogously, we can construct the morphism S'' such that $S' \circ S'' = \mathrm{id}_N$. Using the associativity law we obtain: $S = S \circ (S' \circ S'') = (S \circ S') \circ S'' = S''$ thus $S^{-1} := S'$ is the inverse morphism of S. $\square$

We also need the following technical lemma:

Lemma 6.3.9. *Let M be a representation of a triangular differential biquiver (Q, ∂), let $\{N_i | i \in I\}$ be a set of vector spaces of dimension $\dim(N_i) = \dim(M_i)$ and let $\{S_i : M_i \to N_i, S(v) : M_i \to N_j | i, j \in I, v \in Q_1(i, j)\}$ be a set of linear maps, with invertible S_i.*

Then there exists a representation $N \in \mathsf{Rep}(Q, \partial)$ and a morphism of representations $S : M \to N$, with the given set of vector spaces and the given set of linear maps.

Proof. To prove the lemma we define morphisms $N(a) : N_i \to N_j$ for each $a \in Q_0(i, j)$ in such a way that relations (6.11) hold. For a minimal edge $a \in Q_0(i, j)$, $\partial(a) = 0$ define $N(a) := S_j M(a) S_i^{-1}$. For any other arrow a define $N(a)$ by induction on level. Assume $N(b)$ is defined for all b of level smaller than that of a. Recall that if $\partial(a) = \sum \alpha p_1 \cdot v \cdot p_2$, where $\alpha \in \Bbbk$, p_1 and p_2 are solid paths consisting of arrows of level smaller than level of a, and $p_1 \cdot v \cdot p_2$ is a path from i to j of degree one; then $S(\partial(a)) = \sum \alpha N(p_1)S(v)M(p_2)$. Define $N(a) = S_j M(a) S_i^{-1} + S(\partial(a))S_i^{-1}$. That completes the proof. $\square$

Theorem 6.3.10 ([KR75]). *If (Q, ∂) is a triangular differential biquiver then $Rep(Q, \partial)$ is karoubian.*

Proof. Let $M \in \mathsf{Rep}(Q, \partial)$ be a representation and $E : M \to M$, $E^2 = E$ be a nontrivial idempotent. We claim that M is decomposable. Indeed, consider the set of linear maps: $S_i := \mathrm{id}_{M_i}$ for $i \in I$ and $S(v) := E_j E(v) - E(v) E_i$ for $v \in Q_1(i, j)$. Then by Lemma 6.3.9 there exists a representation N and an isomorphism $S : M \to N$. Note that $S_i^{-1} = \mathrm{id}_{M_i}$ and $S^{-1}(v) = -S(v)$. Define an endomorphism $E' := S \circ E \circ S^{-1}$ of N :

$$\begin{tikzcd} M \arrow[r, "E"] & M \arrow[d, "S"] \\ N \arrow[u, "S^{-1}"] \arrow[r, "E'"] & N, \end{tikzcd}$$

then $E'_i = E_i : M_i \to M_i$ and for $v \in Q_1(i,j)$ with $\partial(v) = 0$ we have

$$\begin{aligned}
E'(v) &= \big(S \circ E \circ S^{-1}\big)(v) \\
&= \big((S \circ E) \circ S^{-1}\big)(v) \\
&= (S \circ E)_j S^{-1}(v) + (S \circ E)(v) S_i^{-1} \\
&= -(S \circ E)_j S(v) + (S \circ E)(v) S_i^{-1} \qquad (\text{since } S^{-1}(v) = -S(v)) \\
&= -E_j S(v) + (S \circ E)(v) \qquad (\text{since } S_i = S_i^{-1} = id_{M_i}) \\
&= -E_j S(v) + S_j E(v) + S(v) E_i \\
&= -E_j\big(E_j E(v) - E(v) E_i\big) + E(v) + \big(E_j E(v) - E(v) E_i\big) E_i \\
&= -E_j E(v) + E_j E(v) E_i + E(v) + E_j E(v) E_i - E(v) E_i \\
&= E(v) - E_j E(v) - E(v) E_i + 2 E_j E(v) E_i \\
&= 0.
\end{aligned}$$

The last equality should be explained. Indeed, since E is an idempotent, $E(v) = E^2(v) = E_j E(v) + E(v) E_i$ and multiplying this equality by E_j from the left and by E_i from the right we obtain: $E_j E(v) E_i = 0$.

The endomorphism $E' : N \to N$ is a nontrivial idempotent $E'^2 = E'$ with $E'(v) = 0$ and $E'_i = E_i$, i.e. E' is a morphism not only in $\mathsf{Rep}(Q, \partial)$ but also in the category of representations of the solid quiver Q_0. Hence, the representation N splits $N = \ker(E') \oplus \operatorname{im}(E')$. Induction on level completes the proof. □

A normal free triangular bocs is called a *Roiter bocs*. From now on we assume all differential biquivers to be triangular. Finally we are ready to give the definition, which is the underlying reason for introducing the formalism.

Definition 6.3.11. A *matrix problem* is the problem of describing the category $ind(\mathsf{Rep}(Q, \partial))$, where (Q, ∂) is a triangular differential biquiver.

Remark 6.3.12. In Section 6.6 we modify this definition and consider matrix problems with respect to *bricks*, i.e. representations with no nonscalar endomorphisms.

One remark on matrix problems

Sometimes it is useful to come back to the original informal definition. Let (Q, ∂) be a differential biquiver, then its representation M can be considered as a matrix in some general sense: it is a matrix divided into $n \times n$ blocks and if $a : i \to j$ is a solid arrow, then $M(a)$ is a block on the entry (ji). If there are more than one arrow from i to j, then there are several blocks placed on the entry (ji). A morphism $S : M \to N$ is also a generalized matrix S consisting of blocks $S(v)$ on places (ji) for each dotted arrow $v : i \to j$ and blocks $S_i := S(w_i)$ on the diagonal together with a block matrix equation

$$S *_\partial M = N *_\partial S, \tag{6.15}$$

where the multiplication $*_\partial$[3] of such generalized matrices is the set of equations for blocks of M determined by the differential ∂ according to formula (6.11), and the composition $S *_\partial T$ of $S : M \to N$ and $T : N \to L$ is defined according to (6.12).

Note that a morphism $S : M \to N$ is an isomorphism if S_i are isomorphisms, for all $i \in I$. In this case keeping with matrix terminology S determines a *transformation* of N

$$N \mapsto S^{-1} *_\partial N *_\partial S, \tag{6.16}$$

where S^{-1} is the transformation inverse to S, the multiplication is applied correctly since S^{-1} has the same block form as S.

Example 6.3.13. In Example 6.3.4 the matrix $M = \big(M(a), M(b)\big)$ can be written in the form

$$\begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 1 & & M(a) & M(b) \\ \hline 2 & & & \\ \hline 3 & & & \\ \hline \end{array}$$

and respectively the matrix S has the form

$$\begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 1 & S_1 & & \\ \hline 2 & & S_2 & \\ \hline 3 & & S(v) & S_3 \\ \hline \end{array},$$

where the empty spaces always denote zero blocks. Thus we obtain the following transformation rule:

$$M = \big(M(a), M(b)\big) \mapsto \big(S_1^{-1} M(a) S_2 - M(b) S(v), S_1^{-1} M(b) S_3\big)$$

or equivalently: $M \mapsto S^{-1} *_\partial M *_\partial S$.

[3] if the biquiver Q contains no double arrows the multiplication $*_\partial$ frequently appears to be just the usual matrix multiplication modulo blocks (ij) for i j such that $A(j, i) = 0$.

Step of matrix reduction

Let us recall the general approach to matrix problems. Let $b : 2 \to 1$ be a minimal edge of a differential biquiver (Q, ∂) Then to describe its representations we reduce the matrix $M(b)$ by the Gauß algorithm to the form $\left(\begin{smallmatrix} 0 & 0 \\ \mathbb{I} & 0 \end{smallmatrix}\right)$. Substituting this form in (6.15) we obtain some restrictions for $S(w_1)$, $S(w_2)$ and probably for some other blocks, what implies a finer structure for S. Moreover, if b is involved in any differential $\partial a = vb + ...$, then some sub blocks of $M(a)$ can be killed by $S(v)$ and thus $M(a)$ becomes a finer block structure, too. From the nonreduced sub blocks we form a new matrix $M_{(new)}$ and correspondingly from the independent subblocks of S form a new matrix $S_{(new)}$. The new multiplication $*_{\tilde{\partial}}$ is uniquely determined by the old one. Such a procedure is called *a step of matrix reduction* and gives the matrix interpretation of a reduction step for differential biquivers, which we consider rigorously in Section 6.5.

In the following section we list some examples of differential biquivers, which appear as problems of classification of vector bundles on degenerations of elliptic curves.

6.4 Examples of Roiter Bocses

To illustrate the formalism let us reformulate the matrix problems obtained in Section 2.5 in terms of Roiter bocses.

Matrix problem for a nodal curve

In the following two examples we formulate the matrix problem for a nodal curve in terms of differential biquivers.

Example 6.4.1 (Matrix problem with two blocks). For simplicity assume that $\tilde{\mu}$ consists of two blocks 0 and 1, the general case will be treated in Example 6.4.2. Recall that $\tilde{\mu}$ consists of two block matrices $\mu(0)$ and $\mu(\infty)$

$$\tilde{\mu} = \big(\mu(0), \mu(\infty)\big) = \left(\begin{array}{|c|}\hline B_0(0) \\ \hline B_1(0) \\ \hline \end{array}\, , \begin{array}{|c|}\hline B_0(\infty) \\ \hline B_1(\infty) \\ \hline \end{array} \begin{array}{c} {\scriptstyle 0} \\ {\scriptstyle 1} \end{array} \right)$$

and transformations are pairs $(\bar{F}, f)$, where F is of the form

$$\bar{F} = (F(0), F(\infty)) = \left(\begin{array}{|cc|}\hline S_0 & 0 \\ \hline S_{10}^{0} & S_1 \\ \hline \end{array}\, , \begin{array}{|cc|}\hline S_0 & 0 \\ \hline S_{10}^{\infty} & S_1 \\ \hline \end{array} \begin{array}{c} {\scriptstyle 0} \\ {\scriptstyle 1} \end{array} \right)$$

and f is given by an invertible matrix S together with relations (2.17):

$$\big(\mu(0), \mu(\infty)\big) \mapsto \big(F(0)^{-1}\mu(0)S, F(\infty)^{-1}\mu(\infty)S\big),$$

Consider the biquiver:

$$\begin{tikzcd}
0 \arrow[dd, dotted, bend right, "u^\infty"'] \arrow[dd, dotted, "u^0"] & \\
& * \arrow[ul, bend right, "a^\infty"'] \arrow[ul, "a^0"'] \arrow[dl, "b^0"] \arrow[dl, bend left, "b^\infty"] \\
1 &
\end{tikzcd} \tag{6.17}$$

with the differential:

$$\begin{aligned}
\partial(a^0) &= \partial(a^\infty) = \partial(u^0) = \partial(u^\infty) = 0 \\
\partial(b^0) &= u^0 a^0 \\
\partial(b^\infty) &= u^\infty a^\infty
\end{aligned}$$

Then $\tilde{\mu}$ can be considered as a representation M of the biquiver (Q, ∂). Indeed, for $i = 0, \infty$, taking $M(a_i) := B_0(i)$, $M(b^i) := B_1(i)$ and morphisms $M(u^i) := S^i_{10}$, $M(w_0) = S_0$, $M(w_1) = S_1$ and $M(w_*) = S$ we obtain the matrix problem for a node with two blocks.

Example 6.4.2 (General matrix problem). If μ consists of more than two blocks we get the following differential biquiver (Q, ∂) : For a vertex $j \in \{1, \ldots, n\}$ the biquiver Q restricted to $\{c_j, *\}$ is

$$\begin{tikzcd}
* \arrow[r, bend left, "a_j^\infty"] \arrow[r, bend right, "a_j^0"'] & c_j
\end{tikzcd}$$

For any two vertexes c_j and c_k $j, k \in \{1, \ldots, n\}$ and $k < j$ the biquiver Q restricted to $\{c_j, c_k\}$ is

$$\begin{tikzcd}
c_k \arrow[r, dotted, bend left, "u_{jk}^\infty"] \arrow[r, dotted, bend right, "u_{jk}^0"'] & c_j
\end{tikzcd} \tag{6.18}$$

The set of vertices $\{1, \ldots, n\}$ is totally ordered with respect to the relation

(6.18). The differential is defined as follows:

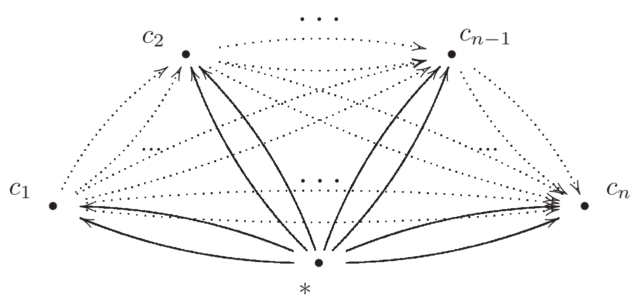
$$\partial a_j^0 = \sum_{k<j} u_{jk}^0 a_k^0$$

$$\partial a_j^\infty = \sum_{k<j} u_{jk}^\infty a_k^\infty$$

$$\partial u_{jk}^0 = \sum_{k<l<j} u_{jl}^0 u_{lk}^0$$

$$\partial u_{jk}^\infty = \sum_{k<l<j} u_{jl}^\infty u_{lk}^\infty$$

Putting all arrows together we obtain a biquiver Q of the form:



Analogously to the previous case, a description of isomorphism classes of representations $M \in \mathsf{Rep}(Q, \partial)$ corresponds to the matrix problem formulated in Subsection 2.5.

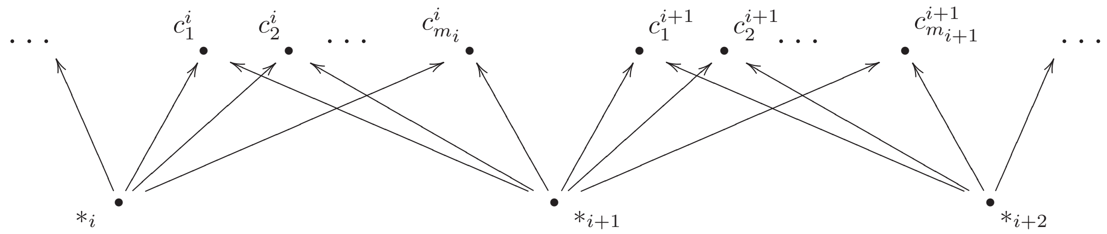
Note that vertices $c_1, \ldots, c_n$ together with the set of dotted arrows either $u^0 := \{u_{jk}^0\}$ or $u^\infty := \{u_{jk}^\infty\}$ form a totally ordered set.

Matrix problem for cycles of projective lines

In the following example we reformulate the matrix problem for vector bundles on cycles of projective lines in terms of Roiter bocses.

Example 6.4.3. For cycles of projective lines we construct a biquiver similarly to the previous example. The set of vertexes consists of n series of $c^i = \{c_1^i, \ldots c_{m_i}^i\}$ and n stars $*_1, \ldots, *_n$. The quiver Q restricted to a set c^i consists of two parallel dotted arrows $(u_{jk}^i)^0$ and $(u_{jk}^i)^0$ for $k < j < m_i$ as shown in (6.18). The solid arrows are $a_{ji}^0 : *_i \longrightarrow c_j^i$ and $a_{ji}^\infty : *_{i+1} \longrightarrow c_j^i$, for $i \in \{1, \ldots n-1\}$, $j \in \{1, \ldots, m_i\}$ and $a_{jn}^0 : *_n \longrightarrow c_j^n$ and $a_{jn}^\infty : *_1 \longrightarrow c_j^n$,

$j \in \{1, \ldots, m_n\}$. The solid quiver Q_0 has form



The biquiver Q sometimes is called a *bunch of chains,* Since the sets c^i are totally ordered sets and $Q_0|_c^i$ can be considered as two chains of dotted arrows with the transitivity relation. The differential is defined as follows:

$$\partial a_{ji}^0 = \sum_{k<j} (u_{jk}^i)^0 a_{ki}^0,$$
$$\partial a_{ji}^\infty = \sum_{k<j} (u_{jk}^i)^\infty a_{ki}^\infty.$$
$$\partial u_{jk}^{0i} = \sum_{k<l<j} u_{jl}^{0i} u_{lk}^{0i}$$
$$\partial u_{jk}^{\infty i} = \sum_{k<l<j} u_{jl}^{\infty i} u_{lk}^{\infty i}$$

Remark 6.4.4. The matrix problems (i.e. differential biquivers) described in Examples 6.4.1–6.4.3 are known in the representation theory as *Gelfand problems* or *representations of bunches of chains.* The isomorphism classes of indecomposable representations of this class of matrix problems were described in [Bon92, KL86, CB89].

Note that, problems of this kind are frequently formulated in the bimodule language (see for example [DG01]). In Appendix C we show that any bimodule problem can be presented as a Roiter bocs with the same biquiver.

Original matrix problem for a cuspidal cubic curve

For simplicity we only consider the matrix problem with two blocks. The general case can be easily deduced along similar lines as in Example 6.4.2.

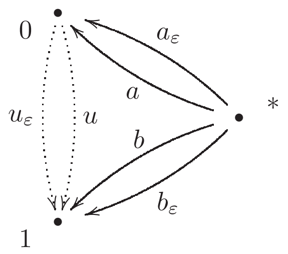
Example 6.4.5 (Matrix problem with two blocks). Assume that $\tilde{\mu}$ consists of two blocks 0 and 1, recall the matrix problem for a cuspidal cubic curve obtained in Section 3:

$$\tilde{\mu} = \mu(0) + \varepsilon \cdot \mu_\varepsilon(0) = \begin{array}{|c|}\hline B_0(0) \\ \hline B_1(0) \\ \hline\end{array} + \varepsilon \cdot \begin{array}{|c|}\hline B_0(0) \\ \hline B_1(0) \\ \hline\end{array} \begin{array}{l} 0 \\ 1 \end{array}$$

and transformations are pairs $(\bar{F}, f)$, where F is of the form

$$\bar{F} = F(0) + \varepsilon \cdot \frac{dF}{dz_0}(0) = \boxed{\begin{array}{cc} S_0 & 0 \\ \hline S_{10} & S_1 \end{array}} + \varepsilon \cdot \begin{array}{c|c} \boxed{\begin{array}{cc} 0 & 0 \\ \hline S'_{10} & 0 \end{array}} & \begin{array}{c} 0 \\ 1 \end{array} \end{array}$$

and f is given by an invertible matrix S together with relations (3.3). This matrix problem corresponds to the category of representations of the following differential biquiver (Q, ∂):



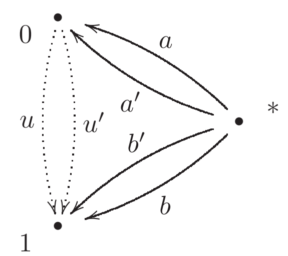
(6.19)

with the differential:

$$\begin{aligned} \partial(a) &= \partial(a_\varepsilon) = 0 \\ \partial(b) &= ua \\ \partial(b_\varepsilon) &= ua_\varepsilon + u_\varepsilon a \end{aligned}$$

Note that the biquiver Q is the same as in diagram (6.17) from Example 6.4.1. The matrix problem formulated in Section 3 boils down to the classification of representations of this biquiver (Q, ∂). Indeed, taking $M(a) := B_0$, $M(a_\varepsilon) := B_{0\varepsilon}$, $M(b) := B_1$ and $M(b_\varepsilon) := B_{1\varepsilon}$ we construct an object, and morphisms are obtained taking $M(u) := S^s_{10}$, $M(u_\varepsilon) := S'_{10}$, $M(w_0) = S_0$, $M(w_1) = S_1$ and $M(w_*) = S$.

Remark 6.4.6. Note that the differential biquiver corresponding to a cuspidal cubic curve is a degeneration of the differential biquiver corresponding to a nodal curve. Indeed, the underlining biquivers in both cases are the same.



Thus the corresponding bocses have isomorphic categories A and isomorphic A-bimodule V. Obviously, the differential should be defined for arrows b and b' only. The space of admissible differentials has dimension 8 over $\Bbbk$. Moreover, two differentials are isomorphic if one can be obtained from the other by recollecting vector bases $\{a, a'\}, \{b, b'\}$ and $\{u, u'\}$. Hence, a differential can be determined by its canonical form. Taking the proper base change, we obtain:

$$\partial(b) = \begin{array}{|cc|l} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{a'} & \\ \cline{1-2} 1 & 0 & u \\ 0 & 0 & u' \\ \cline{1-2} \end{array} \quad \text{and} \quad \partial(b') = \begin{array}{|cc|l} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{a'} & \\ \cline{1-2} 0 & * & u \\ * & * & u' \\ \cline{1-2} \end{array}$$

Consider the following family of differentials:

$$\partial(b) = \begin{array}{|cc|l} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{a'} & \\ \cline{1-2} 1 & 0 & u \\ 0 & 0 & u' \\ \cline{1-2} \end{array} \quad \text{and} \quad \partial(b') = \begin{array}{|cc|l} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{a'} & \\ \cline{1-2} 0 & 1 & u \\ 1 & \lambda & u' \\ \cline{1-2} \end{array}$$

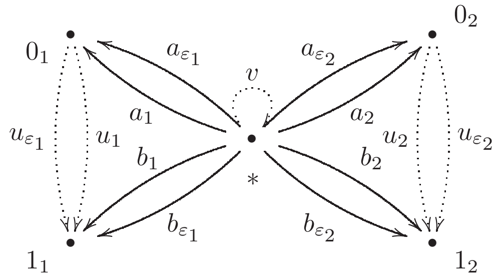
If $\lambda \neq 0$ it can be reduced to $\lambda = 1$, then by transformation $u \mapsto u + u'$ and $a \mapsto a + a'$ we obtain

$$\partial(b) = \begin{array}{|cc|l} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{a'} & \\ \cline{1-2} 1 & 0 & u \\ 0 & 0 & u' \\ \cline{1-2} \end{array} \quad \text{and} \quad \partial(b') = \begin{array}{|cc|l} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{a'} & \\ \cline{1-2} 0 & 0 & u \\ 0 & 1 & u' \\ \cline{1-2} \end{array}$$

which is the differential of the nodal curve: $\partial(b) = ua$ and $\partial(b') = u'a'$. Analogously, if $\lambda = 0$ we obtain the differential for the cuspidal cubic curve: $\partial(b) = ua$ and $\partial(b') = u'a + ua'$.

Original matrix problem for a tacnode curve

Example 6.4.7 (Matrix problem with two blocks). Analogously to the previous cases let us find the differential biquiver (Q, ∂) corresponding to the matrix problem of a tacnode curve, formulated in Chapter 4 (relations (4.3)–(4.6)). The biquiver:



(6.20)

and the differential

$$\begin{aligned} \partial(a_i) &= 0 \\ \partial(a_{\varepsilon_i}) &= a_i v \\ \partial(b_i) &= u_i a_i \\ \partial(b_{\varepsilon_i}) &= u_i a_{\varepsilon_i} + u_{\varepsilon_i} a_i + b_i v. \end{aligned}$$

for $i = 1, 2$.

6.5 Reduction Algorithm

Let (Q, ∂) be a triangular differential biquiver. In this section we introduce two base-change procedures based on Proposition 6.2.5, which transform (Q, ∂) to a Morita-equivalent differential biquiver $(\widetilde{Q}, \tilde{\partial})$. For a sincere representation $M \in \mathsf{Rep}(Q, \partial)$ the procedures *reduce* its norm i.e. for the corresponding representation of $(\widetilde{Q}, \tilde{\partial})$ we have $\|M\| > \|\widetilde{M}\|$.

Definition 6.5.1. A solid arrow $a : i \to j$ of a differential biquiver (Q, ∂) is called *superfluous* or *non-regular* if

$$\partial(a) = \alpha v + \sum_k \beta_k p_k,$$

where $\alpha, \beta_k \in \Bbbk$ are coefficients and $\alpha \neq 0$, and p_k are paths from $i \mapsto j$ of degree one such that the arrow v is not contained in any of them.

Remark 6.5.2. In terms of bocses, an arrow a is superfluous if its differential can be included in a system of generators of $\overline{V}$.

Proposition 6.5.3 (Regularization). *Let a be a superfluous arrow of a triangular differential biquiver (Q, ∂). Let $\widetilde{Q}$ be a biquiver with the same set of vertices, the set of solid arrows $\widetilde{Q}_0 := Q_0 \setminus \{a\}$, the set of dotted arrows $\widetilde{Q}_1 := Q_1 \setminus \{v\}$; and let $\tilde{\partial} : \Bbbk\widetilde{Q} \to \Bbbk\widetilde{Q}$ be a $\Bbbk$-linear map obtained from ∂ by substitution $v = -\alpha^{-1}\big(\sum_k \beta_k p_k\big)$. Then $(\widetilde{Q}, \tilde{\partial})$ is a triangular differential biquiver and the categories $\mathsf{Rep}(Q, \partial)$ and $\mathsf{Rep}(\widetilde{Q}, \tilde{\partial})$ are equivalent.*

Proof. Let $\mathscr{A} := (A, V)$ be a Roiter bocs corresponding to the differential biquiver (Q, ∂). We will show that the set $(\widetilde{Q}, \tilde{\partial})$ is the differential biquiver corresponding to the bocs $\mathscr{A}^B$, where $B := A/(a)$ and $F : A \to B$ is the natural projection. The claim follows from Proposition 6.2.5 if we choose a proper subbox $\mathscr{A}'$, a category B' and a functor $F' : A' \to B'$.

By Remark 6.5.2 we can assume $u := \partial(a)$ to be a generator of $\overline{V}$. Consider the sub-bocs $\mathscr{A}' := (A', V')$, with $A' = \langle a \rangle_{\Bbbk} \subset \mathscr{A}$ and $\overline{V}' = \langle u \rangle_{A'}$ and section $\mathfrak{w}' = \{w_1, w_2\}$ if $a : 1 \to 2$ is an edge and $\mathfrak{w}' = \{1\}$ if $a : 1 \to 1$ is as a loop.

Define $B' := ind(\mathsf{Rep}\,\mathscr{A}')$ to be the discrete category of iso-classes of indecomposable representations of $\mathscr{A}'$.

If $a : 1 \to 2$ is an edge then there are three indecomposable representations of the category A', namely $X_1 : \Bbbk_1 \to 0_2$, $Y : 0_1 \to \Bbbk_2$ and $Z : \Bbbk_1 \xrightarrow{id} \Bbbk_2$, but if we consider representations of the bocs $\mathscr{A}'$, the representation Z becomes decomposable: $Z \cong X_1 \oplus Y$ with isomorphism $S = (S_1 = S_2 = S(v) = \mathrm{id})$. Thus the discrete category B' consists of two elements X_1 and Y.

If $a : 1 \to 1$ is a loop, the indecomposable representations of A' are Jordan blocks $J(\lambda)$, $\lambda \in \Bbbk$. But if we consider representations of the bocs

$\mathscr{A}'$, all representations of the same dimension are isomorphic. Indeed, let $M, N \in \mathsf{Rep}(\mathscr{A}')$ be two representations with $\dim M_1 = \dim N_1$. Construct the isomorphism $S : M \to N$, $S = \{S_1, S(v)\}$, where S_1 is an isomorphism and $S(v) := N(a)S_1 - S_1 M(a)$. Hence, a representation of $\mathscr{A}'$ of dimension greater then one is isomorphic to a decomposable one, and B' contains a unique object $X_1 : \Bbbk_1 \xrightarrow{0} \Bbbk_1$.

Let $F' : A' \to B'$ be the forgetful functor, which "forgets" the morphism a and preserves objects. Then in terms of Proposition 6.2.5 the push-out B in both cases will be isomorphic to the category $A/(a)$. The constructed bocs $\mathscr{A}^B$ is normal, free and triangular i.e. a Roiter bocs. However, the substitution of the arrow v can affect linearity. $\square$

Using Proposition 6.5.3 we can get rid of superfluous arrows. This procedure is called *regularization.* It preserves the vector dimension of a sincere representation M but reduces its norm.

Remark 6.5.4. Since by regularization the linearity condition can be affected, a bimodule problem can degenerate to a problem which is no longer of a bimodule type.

Minimal Edge Reduction

The most essential type of base change procedures, which we use, is called the *minimal edge reduction* proposed by Roiter in [Ro79] to encode the Gauß Algorithm. Apply Proposition 6.2.5 to the following situation. Let $\mathscr{A} = (A, V)$ be a Roiter bocs with a normal section $\mathfrak{w}$ and let (Q, ∂) be the corresponding differential biquiver. Assume $b : 1 \to 2$ to be a solid edge arrow minimal with respect to triangularity (i.e. $\partial(b) = 0$). Let A' be the subcategory generated by b and $\mathscr{A}' = A'$ be a principal bocs with quiver Q':

$$\overset{1}{\bullet} \xrightarrow{\;b\;} \overset{2}{\bullet} \; .$$

As was mentioned above, A' has three indecomposable representations $X : \Bbbk_1 \to 0_2$, $Y : 0_1 \to \Bbbk_2$ and $Z : \Bbbk_1 \to \Bbbk_2$. Let B' be the discrete category $ind(\mathsf{Rep}(\mathscr{A}'))$ with three objects X, Y and Z, which for convenience we rename as $1, 2$ and 0.

The functor $F' : A' \to \mathsf{add}(B')$ is defined as

$$\begin{aligned} 1 &\mapsto \; 0 \oplus 1 \; , \\ 2 &\mapsto \; 2 \oplus 0 \; , \\ b &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \end{aligned}$$

According to Proposition 6.2.5 the bocs $\mathscr{A}^B := (B, {}^B V^B)$ is Morita-equivalent to $\mathscr{A}$, where B is a skeleton of the push-out of the triple $(\mathsf{add}(B') \xleftarrow{F'} A' \xhookrightarrow{i} A)$.

We claim that the bocs $\mathscr{A}^B$ is normal, free and triangular. Moreover, it is linear if so is $\mathscr{A}$. In other words, from the push-out we recover a new differential biquiver $(\widetilde{Q}, \tilde{\partial})$ Morita equivalent to (Q, ∂). Let us have a look at its structure.

1. The set of vertices $\tilde{I}$ consists of all vertices $i \in I$ together with the new one 0:
$$\tilde{I} = \{0\} \cup I.$$

2. Consider the image of the section w under the morphism F :
$$F(w_1) := \begin{pmatrix} \xi_0, \xi_{01} \\ \xi_{10}, \xi_1 \end{pmatrix}$$
and
$$F(w_2) := \begin{pmatrix} \eta_2, \eta_{20} \\ \eta_{02}, \eta_0 \end{pmatrix}.$$
Since $\partial(b) = 0$, we get
$$0 = F(\partial(b)) = F(bw_1 - w_2 b) = F(b)F(w_1) - F(w_2)F(b).$$
In terms of matrices this is equivalent to the equality:
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_0, \xi_{01} \\ \xi_{10}, \xi_1 \end{pmatrix}, = \begin{pmatrix} \eta_2, \eta_{20} \\ \eta_{02}, \eta_0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
from which we deduce $\xi_{20} = \eta_{01} = 0$ and $\eta_0 = \xi_0$. Since w is a normal section, i.e. $\mu(w_i) = w_i \otimes w_i$, by the definition of comultiplication $\tilde{\mu} := \mu_{\mathscr{A}^B}$. For $i \in I$ we have:
$$\tilde{\mu}(F(w_i)) = F(\tilde{\mu}(w_i)) = F(w_i \otimes w_i) = F(w_i) \otimes F(w_i),$$
where F denotes the extension of F on the tensor category $A[\overline{V}]$. For $i = 1$ we obtain
$$\tilde{\mu}(F(w_1)) = \begin{pmatrix} \tilde{\mu}(\xi_0) & 0 \\ \tilde{\mu}(\xi_{10}) & \tilde{\mu}(\xi_1) \end{pmatrix}$$
and
$$F(w_1) \otimes F(w_1) = \begin{pmatrix} \xi_0^2 & 0 \\ \xi_{10} \otimes \xi_0 + \xi_1 \otimes \xi_{10} & \xi_1^2 \end{pmatrix},$$
hence, $\tilde{\mu}(\xi_0) = \xi_0^2$, $\tilde{\mu}(\xi_1) = \xi_1^2$ and $\tilde{\partial}(\xi_{10}) = 0$. The same equalities can be written for entries of w_2. Thus, taking $\tilde{w}_1 := \xi_1$, $\tilde{w}_2 := \eta_2$, $\tilde{w}_0 := \xi_0 = \eta_0$ we have
$$F(w_1) = \begin{pmatrix} \tilde{w}_0 & 0 \\ \xi & \tilde{w}_1 \end{pmatrix}, \text{ and } F(w_2) = \begin{pmatrix} \tilde{w}_2 & 0 \\ \eta & \tilde{w}_0 \end{pmatrix}.$$

Taking $\tilde{w}_i := F(w_i)$ for $i \in I \setminus \{1, 2\}$ we construct a normal section of the bocs $\mathscr{A}^B$.

For simplicity we denote $\xi := \xi_{10}$ and $\eta := \eta_{02}$ and call them the "new" dotted arrows, since ξ and η belong to the kernel of the counit $\overline{{}^B V^B}$, and thus $\xi, \eta \in \widetilde{Q}_1$. As was shown above $\tilde{\partial}(\xi) = \tilde{\partial}(\eta) = 0$.

3. Besides ξ and η, arrows of $\widetilde{Q}_0 \cup \widetilde{Q}_1$ are entries of images $F(x)$ of arrows $x \in Q_0 \cup Q_1$. The morphism $F(x)$ maps $F(i)$ to $F(j)$ and $F(x) = id_{F(j)} \otimes x \otimes id_{F(i)}$. Introduce new notations $\tilde{x}_{ij} := id_{\tilde{i}} \cdot x \cdot id_{\tilde{j}}$ and write $\tilde{x}_i$ instead of $\tilde{x}_{ii}$. Entries $\tilde{x}_{ij}$ are arrows of $\widetilde{Q}$ provided that x is an arrow of Q.

   If $x : i \to j$ is an arrow for $i, j \in I \setminus \{1, 2\}$ then $\tilde{x} := F(x) : \tilde{i} \to \tilde{j}$ itself is an arrow. Otherwise $F(x)$ induces more then one arrow: if $j = 1$ and $i \neq 1, 2$ then
$$F(x) = \begin{pmatrix} id_0 \cdot x \cdot id_{\tilde{i}} \\ id_1 \cdot x \cdot id_{\tilde{i}}, \end{pmatrix} = \begin{pmatrix} \tilde{x}_{0i} \\ \tilde{x}_{1i} \end{pmatrix},$$
   if $j = 2$, $i \neq 1, 2$
$$F(x) = \begin{pmatrix} \tilde{x}_{2i} \\ \tilde{x}_{0i} \end{pmatrix},$$
   if $i = 1$ or $i = 2$ and $j \in I \setminus \{1, 2\}$ then
$$F(x) = (\tilde{x}_{j1}, \tilde{x}_{j0})$$
   or respectively
$$F(x) = (\tilde{x}_{j0}, \tilde{x}_{j2}).$$
   Analogously, for $x : 1 \to 1$
$$F(x) = \begin{pmatrix} \tilde{x}_0 & \tilde{x}_{01} \\ \tilde{x}_{10} & \tilde{x}_1 \end{pmatrix};$$
   for $x : 2 \to 2$
$$F(x) = \begin{pmatrix} \tilde{x}_2 & \tilde{x}_{20} \\ \tilde{x}_{02} & \tilde{x}_0 \end{pmatrix};$$
   for $x : 1 \to 2$
$$F(x) = \begin{pmatrix} \tilde{x}_{20} & \tilde{x}_{21} \\ \tilde{x}_0 & \tilde{x}_{01} \end{pmatrix};$$
   and respectively for $x : 2 \to 1$
$$F(x) = \begin{pmatrix} \tilde{x}_{02} & \tilde{x}_0 \\ \tilde{x}_{12} & \tilde{x}_{10} \end{pmatrix}.$$

4. The differential $\tilde{\partial}$ on arrows $x_{ij} \in \widetilde{Q}_0 \cup \widetilde{Q}_1$ is determined by the equation:
$$F(\partial(x)) = \tilde{\partial}(F(x)).$$
Constructed map $\tilde{\partial}$ is a differential. Indeed, $\tilde{\partial}^2(\tilde{x}) = 0$ for all $\tilde{x} \in \widetilde{Q}_0 \cup \widetilde{Q}_1$, since
$$\tilde{\partial}^2(F(x)) = \tilde{\partial}(\tilde{\partial}(F(x))) = \tilde{\partial}(F(\partial(x))) = F(\partial(x)).$$

5. Now we check the triangularity property. Let $h : Q_0 \cup Q_1 \to \mathbb{N}$ be the level map. Put an order $\tilde{h}$ on arrows $\tilde{x} \in \widetilde{Q}_0 \cup \widetilde{Q}_1$. For $\tilde{x}$ an entry of $F(x)$ and $\tilde{y}$ an entry of $F(y)$, where $x, y \in Q_0 \cup Q_1$ and $h(x) > h(y)$, define $\tilde{h}(\tilde{x}) > \tilde{h}(\tilde{y})$. Let us order arrows $\tilde{x}$ which appear as entries of $F(x)$ for the same arrow $x \in Q_0 \cup Q_1$. Keeping with notations of entries of item (3.) from the definition of $\tilde{\partial}$ and $\tilde{\mu}$ we get

   – for $x : i \to 1$ $\tilde{h}(\tilde{x}_{1i}) > \tilde{h}(\tilde{x}_{0i})$,
   – for $x : 1 \to i$ $\tilde{h}(\tilde{x}_{i0}) > \tilde{h}(\tilde{x}_{i1})$,
   – for $x : i \to 2$ $\tilde{h}(\tilde{x}_{0i}) > \tilde{h}(\tilde{x}_{2i})$,
   – for $x : 2 \to i$ $\tilde{h}(\tilde{x}_{i2}) > \tilde{h}(\tilde{x}_{i0})$

   and

   – for $x : 1 \to 1$ define $\tilde{h}(\tilde{x}_{10}) > \tilde{h}(\tilde{x}_0) = \tilde{h}(\tilde{x}_1) > \tilde{h}(\tilde{x}_{01})$,
   – for $x : 2 \to 2$ define $\tilde{h}(\tilde{x}_{02}) > \tilde{h}(\tilde{x}_0) = \tilde{h}(\tilde{x}_2) > \tilde{h}(\tilde{x}_{20})$,
   – for $x : 2 \to 1$ define $\tilde{h}(\tilde{x}_{12}) > \tilde{h}(\tilde{x}_{02}) = \tilde{h}(\tilde{x}_{10}) > \tilde{h}(\tilde{x}_0)$,
   – for $x : 1 \to 2$ define $\tilde{h}(\tilde{x}_0) > \tilde{h}(\tilde{x}_{01}) = \tilde{h}(\tilde{x}_{20}) > \tilde{h}(\tilde{x}_{21})$.

   Ordered in such a way arrows are involved in differentials of arrows of higher level. Thus the differential biquiver $(\widetilde{Q}, \tilde{\partial})$ is triangular or, in other words, $\mathscr{A}^B$ is a Roiter bocs.

6. Assume that (Q, ∂) is a linear bocs. Hence, if $x \in Q_0(i, j)$ is a dotted arrow with the differential
$$\partial(x) = \sum_{i \dashrightarrow j} \alpha_0 v_0 + \sum_{i \dashrightarrow k \to j} \alpha_1 a_1 v_1 + \sum_{i \to k \dashrightarrow j} \alpha_2 v_2 a_2,$$
where $\alpha_0, \alpha_1, \alpha_2 \in \Bbbk$, $a_1 \in Q(k, j)$, $a_2 \in Q_0(i, k)$ and $v_0 \in Q(i, j)$, $v_1 \in Q_1(i, k)$, $v_2 \in Q_1(k, j)$ then we get an equation
$$F(x)F(w_i) - F(w_j)F(x) = \sum_{i \dashrightarrow j} \alpha_0 F(v_0) + \sum_{i \dashrightarrow k \to j} \alpha_1 F(a_1)F(v_1) + \sum_{i \to k \dashrightarrow j} \alpha_2 F(v_2)F(a_2). \quad (6.21)$$

Rewritten for entries this equation implies formulas for $\tilde{\partial}(\tilde{x})$. Similarly, consider dotted arrows. Assume $x \in Q_1(i,j)$ and

$$\partial(x) = \sum_{i \dashrightarrow k \dashrightarrow j} \alpha u \otimes v,$$

where $\alpha \in \Bbbk$, $u \in Q_1(k,j)$ and $v \in Q_1(i,k)$, then

$$F(x)F(w_i) - F(w_j)F(x) = \sum_{i \dashrightarrow k \dashrightarrow j} \alpha F(u)F(v) \tag{6.22}$$

From equations (6.21),(6.22) obviously follows the linearity property for the differential $\tilde{\partial}$.

Remark 6.5.5. If M is a sincere at vertices 1 and 2 representation of $\mathscr{A}$ then this procedure reduces not only its norm but also its dimension. Indeed, if M is a representation of a bocs A and $\widetilde{M}$ is a representation of the bocs $\mathscr{A}^B$ such that $F^*(\widetilde{M}) = M$ then $\widetilde{M}(i) = M(i)$ for $i \in I \setminus \{1,2\}$ and $M(1) = \widetilde{M}(1 \oplus 0)$ and $M(2) = \widetilde{M}(0 \oplus 2)$. Thus $\dim(M) = \dim(\widetilde{M}) + \dim(\widetilde{M}(0))$ and if $\dim(\widetilde{M}(0)) > 0$ then $\dim(\widetilde{M}) < \dim(M)$. Hence, starting with a representation M in the course of reduction we make its dimension smaller. Thus, the regularization procedure and minimal edge reduction make possible the induction on the norm on all Roiter bocses, as long as there exists a minimal edge or a superfluous arrow. The difficulty appears for bocses, all of whose minimal arrows are loops. In [Dro79] Drozd introduced a partial loop reduction procedure but we do not discuss it here.

Let us illustrate the reduction algorithm on the following example:

Example 6.5.6. Let (Q,∂) be a differential biquiver corresponding to the matrix problem from Subsection 3.2. Then the category $\mathsf{Rep}(Q,\partial)$ is equivalent to the matrix category BM_P constructed for a cuspidal cubic curve.

The corresponding biquiver Q is:

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[loop left, "a_1"] \arrow[r, dotted, bend right, "u_{21}"'] & \overset{2}{\bullet} \arrow[l, bend right, "a_{12}"'] \arrow[loop right, "a_2"] \end{tikzcd} \tag{6.23}$$

with the differential ∂ defined as:

$$\begin{aligned} \partial(a_{12}) &= 0, \\ \partial(a_1) &= a_{12}u_{21} \\ \partial(a_2) &= -u_{21}a_{12}. \end{aligned}$$

Next, we apply the minimal edge reduction to the minimal arrow a_{12}. Recall that

$$F(a_{12}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$F(a_1) = \begin{pmatrix} \tilde{a}_1 & \tilde{a}_{10} \\ \tilde{a}_{01} & \tilde{a}_0 \end{pmatrix}$$

and

$$F(a_2) = \begin{pmatrix} \tilde{a}_0' & \tilde{a}_{02} \\ \tilde{a}_{20} & \tilde{a}_2 \end{pmatrix}.$$

We proceed analogously for dotted arrows and the section:

$$F(u_{21}) = \begin{pmatrix} \tilde{u}_{01} & \tilde{u}_0 \\ \tilde{u}_{21} & \tilde{u}_{20} \end{pmatrix}.$$

Let $\tilde{\partial}$ denote the differential induced by the section $\tilde{w}$, then we should check the relations $\tilde{\partial}(F(a_i)) = F(\partial a_i)$ for $i = 1, 2$:

$$F(a_1)F(w_1) - F(w_1)F(a_1) = F(a_{12})F(u_{21})$$

and

$$F(a_2)F(w_2) - F(w_2)F(a_2) = -F(u_{21})F(a_{12}).$$

Thus we obtain a new differential $\tilde{\partial}$ defined by the formulas:

$$\begin{aligned} \tilde{\partial}(a_0) &= -\tilde{u}_0 - \xi\tilde{a}_{10} \\ \tilde{\partial}(a_{01}) &= -\tilde{u}_{01} - \xi\tilde{a}_0 + \tilde{a}_1\xi \\ \tilde{\partial}(a_{10}) &= 0 \\ \tilde{\partial}(a_1) &= \tilde{a}_{10}\xi. \end{aligned}$$

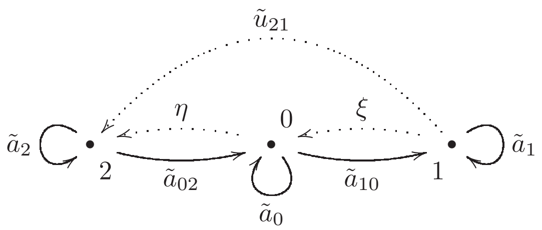
From the second equation we get:

$$\begin{aligned} \tilde{\partial}(a_2) &= -\eta\tilde{a}_{02} \\ \tilde{\partial}(a_{20}) &= \tilde{u}_{20} - \eta\tilde{a}_0' + \tilde{a}_2\eta \\ \tilde{\partial}(a_{02}) &= 0 \\ \tilde{\partial}(a_0) &= \tilde{u}_0 + \tilde{a}_{02}\eta. \end{aligned}$$

Applying the regularization procedure to the arrows $\tilde{a}_{01}$, $\tilde{a}_{20}$ and either to a_0 or a_0' we kill these arrows and simultaneously we get rid of the corresponding dotted arrows: $\tilde{u}_{01}$, $\tilde{u}_{20}$ and one of u_0, since they become dependent and can be expressed by the other arrows. Since $\tilde{u}_0$ appears in the differential of both $\tilde{a}_0$ and $\tilde{a}_0'$, thus substituting its value to the remaining differential we get

$$\tilde{\partial}(a_0) = \tilde{a}_{02}\eta - \xi\tilde{a}_{10}.$$

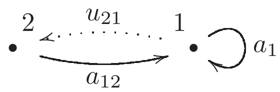
Finally, after a minimal edge reduction and regularization we obtain a differential biquiver $(\widetilde{Q}, \tilde{\partial})$:



(6.24)

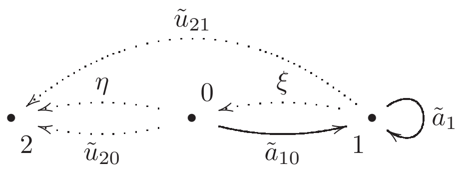
with the differential: $\tilde{\partial}(\tilde{a}_1) = \tilde{a}_{10}\xi$, $\tilde{\partial}(\tilde{a}_2) = -\eta\tilde{a}_{02}$ and $\tilde{\partial}(\tilde{a}_0) = -\xi\tilde{a}_{10} + \tilde{a}_{02}\eta$. The new minimal arrows are now $\tilde{a}_{02}$ and $\tilde{a}_{10}$. Note that if we omit one of the objects 1 or 2, we obtain the same biquiver to that one we started with.

Example 6.5.7. Analogously to Example 6.5.6 starting with the differential biquiver (Q, ∂)



(6.25)

equipped with the differential ∂ given by the formulas: $\partial(a_{12}) = 0$ and $\partial(a_1) = a_{12}u_{21}$. After performing the minimal edge reduction we obtain the biquiver $\widetilde{Q}$



(6.26)

with the differential

$$\tilde{\partial}(\tilde{a}_{10}) = 0$$
$$\tilde{\partial}(\tilde{a}_1) = \tilde{a}_{10}\xi.$$

Restricting $\widetilde{Q}$ onto the set of objects $\{0, 1\}$ we obtain the same problem to that one we started with.

6.6 Bricks

Definition 6.6.1. A representation of a differential biquiver is called a *brick* if it has no nonscalar endomorphisms. Let $\mathsf{Br}(Q, \partial)$ denote the full subcategory of bricks of $\mathsf{Rep}(Q, \partial)$.

Lemma 6.6.2. *Let (Q,∂) be a differential biquiver containing a dotted arrow $u : i \dashrightarrow j$, which is not involved in the differential of a solid arrow. Then* $\mathsf{Rep}(Q)$ *contains no sincere bricks and*

$$\mathsf{Br}(Q,\partial) = \mathsf{Br}(Q^i,\partial) \cup \mathsf{Br}(Q^j,\partial),$$

where (Q^i,∂) is the differential biquiver (Q,∂) restricted to $I\setminus\{j\}$, and respectively, (Q^j,∂) is (Q,∂) restricted to $I\setminus\{i\}$.

Proof. Assume that M is a sincere representation and define an endomorphism $S : M \to M$ as follows: take $S(u)$ to be a nonzero linear map and for all dotted arrows $v \neq u$ put $S(v) := 0$ and S_i to be the identity maps for all $i \in I$. Thus we construct a nonscalar endomorphism of M. Hence, there exist no sincere bricks, and any brick belongs either to $\mathsf{Rep}(Q^i,\partial)$ or to $\mathsf{Rep}(Q^j,\partial)$ or to both. □

Example 6.6.3. Consider the differential biquiver (6.24) obtained in Example 6.5.6 after the minimal edge reduction and regularization. The dotted arrow $\tilde{u}_{21}$ is not involved in any differential, applying Lemma 6.6.2 we get:

$$\mathsf{Br}(Q,\partial) \cong \mathsf{Br}(\widetilde{Q},\tilde{\partial}) = \mathsf{Br}(\widetilde{Q}|_{\{0,1\}},\tilde{\partial}) \cup \mathsf{Br}(\widetilde{Q}|_{\{0,2\}},\tilde{\partial}).$$

As it was mentioned before both biquivers $(\widetilde{Q}|_{\{0,1\}},\tilde{\partial})$ and $(\widetilde{Q}|_{\{0,2\}},\tilde{\partial})$ are equal to the differential biquiver (Q,∂) we started with. Thus the problem from Example 6.5.6 is *self-reproducing* with respect to bricks.

Example 6.6.4. Applying Lemma 6.6.2 for the biquiver $(\widetilde{Q},\tilde{\partial})$ obtained in Example 6.5.7 we get

$$\mathsf{Br}(Q,\partial) \cong \mathsf{Br}(\widetilde{Q},\tilde{\partial}) = \mathsf{Br}(\widetilde{Q}|_{\{0,1\}},\tilde{\partial}) \cup \{2\}.$$

6.7 On one class of brick-tame bocses

In this section we generalize the differential biquivers from Examples 6.5.6 and 6.5.7 and introduce a class BT of brick-tame differential biquivers.

Definition 6.7.1. We say (Q,∂) is a *full* BT-*differential biquiver* if there exists a set of *distinguished* loops:

$$\mathfrak{a} := \{a_i \in Q_0(i,i) | i \in I\}$$

and an injective map:

$$v : Q_0 \setminus \mathfrak{a} \hookrightarrow Q_1,$$

mapping a solid arrow $a : i \to j$ to an opposite directed dotted arrow $v_a := v(a) : j \dashrightarrow i$, and for each distinguished loop $a_i \in \mathfrak{a}$ we get

$$\partial a_i = \sum_{j\to i} b_{ij} v(b_{ij}) - \sum_{i\to j} v(b_{ji}) b_{ji}.$$

A differential biquiver obtained from a full BT-differential biquiver by eliminating some distinguished loops is called a BT-differential biquiver.

In terms of bocses it means the bocs $\mathscr{B}$ is of type BT if $\mathscr{B} = \mathscr{A}^{\bar{A}}$, where $\mathscr{A} = (A, V)$ is a bocs corresponding to a full BT-differential biquiver and $\bar{A} = A/\langle a_{i_1}, \ldots, a_{i_k} | a_{i_j} \in \mathfrak{a}\rangle$. Clearly, any BT-differential biquiver can be uniquely extended to a full one. The differential biquivers (Q, ∂) and $(\widetilde{Q}, \tilde{\partial})$ from Example 6.5.6 are full BT-biquivers. The differential biquivers (Q, ∂) and $(\widetilde{Q}, \tilde{\partial})$ from Example 6.5.7 are also of type BT but not full.

Remark 6.7.2. Note that if (Q, ∂) is a BT-differential biquiver then so is $(Q|_{I'}, \partial)$, for any subset $I' \subset I$, and if (Q, ∂) is full then so is $(Q|_{I'}, \partial)$. In terms of bocses it means: if $\mathscr{A} = (A, V)$ is a Roiter bocs corresponding to (Q, ∂) then we have a morphism of $\mathbb{k}$-algebras: $F : A \to \bar{A}$, where $\bar{A} := A/\langle e_{i_1}, \ldots, e_{i_k} | i_j \in I, i_j \notin I'\rangle$, and the induced morphism of bocses: $F : \mathscr{A} \to \bar{\mathscr{A}}$, where $\bar{\mathscr{A}} = \mathscr{A}^{\bar{A}}$. Obviously, the induced functor $F^* : \mathsf{Rep}(\bar{\mathscr{A}}) \to \mathsf{Rep}(\mathscr{A})$ is fully faithful.

Note that if $\mathbb{s} \in \mathbb{N}^n$ is a vector dimension such that $s_i = 0$ for all $i \notin I'$, then

$$\mathsf{Rep}(Q, \partial)(\mathbb{s}) \cong \mathsf{Rep}(Q|_{I'}, \partial)(\tilde{\mathbb{s}}),$$

where $\tilde{\mathbb{s}} := \mathbb{s}|_{I'}$ is the restriction of $\mathbb{s}$ to I'. This simple consideration allows us to reduce the number of vertices in the course of reduction.

Lemma 6.7.3. *Let (Q, ∂) be a BT-differential biquiver, b be a minimal solid arrow, Then the corresponding dotted arrow v_b is not involved in $\partial(r)$, for any $r \in Q_0 \setminus \{\mathfrak{a}\}$.*

Proof. We start with the following consideration: there are no solid non-distinguished arrows $x \in Q_0(i, j)$, such that for the arrow v_x we have

$$\partial(v_x) = v_b \cdot p + \varphi,$$

where p is a path of degree one and φ is a linear combination of paths of degree two, which are different from $v_b \cdot p$. Indeed, assume otherwise and consider

$$\begin{aligned}\partial^2(a_i) = \quad & -\underline{v_b \cdot p \cdot x} - \varphi \cdot x + v_x \cdot \partial(x) \\ & + \sum_{\cdot \to i} \Big(\partial(m) \cdot v_m + m \cdot \partial(v_m)\Big) \\ & - \sum_{i \to \cdot} \Big(\partial(v_n) \cdot n - v_n \cdot \partial(n)\Big).\end{aligned}$$

The underlined term can be neutralized only by a term of the form $v_n \cdot \partial(n)$, however then $n = b$ and $\partial(b) = 0$. That implies the claim.

Assume that there exists a non-distinguished arrow $r : i \to j$, such that

$$\partial(r) = -q \cdot v_b \cdot p + \phi,$$

where q and p are some paths of degree zero, and ϕ is a linear combination of paths different form $q \cdot v_b \cdot p$. In other words, the biquiver Q contains the following fragment:

$$\begin{tikzcd}
\bullet \arrow[r, dotted, "v_b"'] & \bullet \arrow[d, squiggly, "q"] \\
i \arrow[u, squiggly, "p"] \arrow[r, "r"'] & j
\end{tikzcd}$$

Check the property $\partial^2 = 0$ for the distinguished loop a_j :

$$\begin{aligned}
\partial^2(a_j) &= \partial\Big(r \cdot v_r + \sum_{\substack{m:\,\cdot\to j\\ m\neq r}} m \cdot v_m - \sum_{n:\,j\to\cdot} v_n \cdot n\Big) \\
&= \underline{q \cdot v_b \cdot p \cdot v_r} + \phi \cdot v_r + r \cdot \partial(v_r) \\
&\quad + \sum_{\substack{\cdot\to j\\ m\neq r}} \Big(\partial(m) \cdot v_m + m \cdot \partial(v_m)\Big) \\
&\quad - \sum_{j\to\cdot} \Big(\partial(v_n) \cdot n - v_n \cdot \partial(n)\Big).
\end{aligned}$$

We claim that the underlined term $q \cdot v_b \cdot p \cdot v_r$ cannot be neutralized by any other one. Indeed, the only candidates are that of the form $m \cdot \partial(v_m)$. (The summand $v_n \cdot \partial(n)$ cannot kill the underlined term, since then $n = b$ and $\partial(b) = 0$.) Assume that there is a solid arrow $m_1 : j_1 \to j$ such that $q = m_1 \cdot q_1$ and $\partial(v_m) = q_1 \cdot v_b \cdot p \cdot v_r + \phi_1$, where ϕ_1 is a linear combination of paths different from $q_1 \cdot v_b \cdot p \cdot v_r$. Then consider the loop a_{j_1}

$$\begin{aligned}
\partial^2(a_{j_1}) &= \underline{q_1 \cdot v_b \cdot p \cdot v_r \cdot m_1} + \phi_1 m_1 + v_{m_1}\partial(m_1) \\
&\quad + \sum_{\cdot\to j_1} \Big(\partial(m) \cdot v_m + m \cdot \partial(v_m)\Big) \\
&\quad - \sum_{\substack{j_1\to\cdot\\ n\neq m_1}} \Big(\partial(v_n) \cdot n - v_n \cdot \partial(n)\Big).
\end{aligned}$$

The underlined term $q_1 \cdot v_b \cdot p \cdot v_r \cdot m_1$ can be neutralized only by a term of the form $m \cdot \partial(v_m)$. Assume that there exists an arrow $m_2 : j_2 \to j_1$ such that $q_1 = m_2 \cdot q_2$ and $\partial(v_{m_2}) = q_2 \cdot v_b \cdot p \cdot v_r \cdot m_1 + \phi_2$, where ϕ_2 is a linear combination of paths different from $q_2 \cdot v_b \cdot p \cdot v_r \cdot m_1$. Let k be the length of the path q.

Proceeding this way on the k-th step we get that there exists $m_k : j_k \to j_{k-1}$ such that $q_{k-1} = m_k$ and $\partial(v_{m_k}) = v_b \cdot p \cdot v_r \cdot m_1 \cdot m_2 \dots m_{k-1} + \phi_k$. But this case was considered above. We have seen that there is no solid non-distinguished arrows $x \in Q_0$, such that for the arrow $v_x = v(x)$ we have $\partial(v_x) = v_b \cdot p + \varphi$, where p is a path of degree one and φ is a path of degree two. That implies the proof. □

The condition that the dotted arrow v_b corresponding to a minimal arrow b is not involved in any differential besides distinguished loops, implies several nice properties. Among others it guarantees the existence of a minimal edge arrow as long as a reduction procedure is required.

Lemma 6.7.4. *Let (Q, ∂) be a BT-differential biquiver and $b \in Q_0(1,1)$ be a minimal loop.*

- *If $b = a_1$ is distinguished then $\Bbbk Q = \Bbbk[b]$,*
- *otherwise, there are no sincere bricks and $\mathsf{Br}(Q, \partial) = \mathsf{Br}(Q|_{I\setminus\{1\}}, \partial)$.*

Proof. The first statement follows immediately from the definition of a BT differential biquiver. If $b \neq a_1$ then

$$\partial(a_1) = bv_b - v_b b + \sum_{\substack{j\to i \\ b_{ij}\neq b}} b_{ij} v(b_{ij}) - \sum_{\substack{i\to j \\ b_{ji}\neq b}} v(b_{ji}) b_{ji}$$

and by Lemma 6.7.3 the dotted loop v_b is not involved in any other differential of a solid arrow. We claim that a representation sincere at the vertex 1 is not a brick. For $M \in \mathsf{Rep}(Q, \partial)$ of a vector dimension $(s_1, \dots, s_n)$, with $s_1 > 0$, taking $S_i := \mathbb{I}_{s_i}$ to be the identity map, $S(v) = 0$ for $v \neq v_b$ and $S(v_b) = \mathbb{I}_{s_1}$, we construct a non-scalar endomorphism. That completes the proof. □

Remarks on minimal edge reduction for BT-differential biquivers

Remark 6.7.5. Let (Q, ∂) be a BT-differential biquiver and $b : 2 \to 1$ be a minimal edge. If at least one of the vertices $i = 1, 2$ possesses a distinguished loop, then the differential biquiver $(\widetilde{Q}, \tilde{\partial})$ obtained in the course of minimal edge reduction by the arrow b contains the superfluous arrows $\tilde{a}_{01}$, $\tilde{a}_{20}$ $\tilde{a}_0^1$ and $\tilde{a}_0^2$. The arrow $\tilde{u}_{21} = v_b$ gives rise to the dotted arrows $\tilde{u}_{01}$, $\tilde{u}_{20}$ and $\tilde{u}_0$ such that

$$\begin{aligned} \partial(\tilde{a}_{01}) &= \tilde{u}_{01} + \dots, \\ \partial(\tilde{a}_{20}) &= \tilde{u}_{20} + \dots, \\ \partial(\tilde{a}_0^1) &= \tilde{u}_0 + \dots, \\ \partial(\tilde{a}_0^2) &= \tilde{u}_0 + \dots. \end{aligned}$$

By Lemma 6.7.3, arrows $\tilde{u}_{01}$, $\tilde{u}_{20}$ $\tilde{u}_0^1$ and $\tilde{u}_0^2$ are not involved in a differentials of any other solid arrow. By the regularization procedure we can "kill" the arrows $\tilde{a}_{01}$, $\tilde{a}_{20}$ and either one of the arrows $\tilde{a}_0^1$ and $\tilde{a}_0^2$ together with the corresponding dotted arrows as was done in Example 6.5.6. If $\partial(c) = vb$ for some arrows $c \in Q_0$ and $v \in Q_1$ then $(\widetilde{Q}, \tilde{\partial})$ contains a superfluous arrow $\tilde{c}_0$ with $\partial(\tilde{c}_0) = \tilde{v}_0 + ...$, hence $\tilde{c}_0$ should be also regularized. Some properties of $(\widetilde{Q}, \tilde{\partial})$ are collected in the following proposition.

Proposition 6.7.6. *Let (Q, ∂) be a* BT*-differential biquiver and let $(\widetilde{Q}, \tilde{\partial})$ be the differential biquiver obtained from (Q, ∂) by the minimal edge reduction on the minimal edge $b : 2 \to 1$. Then*

i) $(\widetilde{Q}, \tilde{\partial})$ *is also of type* BT;

ii) if a_i is a distinguished loop in (Q, ∂) then there exists a distinguished loop $\tilde{a}_i$ of $(\widetilde{Q}, \tilde{\partial})$;

iii) if (Q, ∂) contains both distinguished loops a_1 and a_2 then $(\widetilde{Q}, \tilde{\partial})$ contains the distinguished loop $\tilde{a}_0$;

iv) $\mathsf{Br}(Q, \partial) \cong \mathsf{Br}(\widetilde{Q}, \tilde{\partial}) = \mathsf{Br}(\widetilde{Q}^1, \tilde{\partial}) \cup \mathsf{Br}(\widetilde{Q}^2, \tilde{\partial})$, *where* $(\widetilde{Q}^1, \tilde{\partial}) := (\widetilde{Q}|_{\tilde{I}\setminus\{2\}}\tilde{\partial})$ *and respectively* $(\widetilde{Q}^2, \tilde{\partial}) := (\widetilde{Q}|_{\tilde{I}\setminus\{1\}}\tilde{\partial})$.

Proof. A solid arrow $\tilde{a}_{ij} \in \widetilde{Q}_0$ appears as an entry in the matrix $F(a)$ of some $a \in Q_0$. Since (Q, ∂) is a BT-differential biquiver, there exists a unique dotted arrow $v_a := v(a)$, and $F(v_a)$ contains an entry $(\tilde{v}_a)_{ij} := \big(F(v_a)\big)_{ij}$. Define the map

$$\begin{aligned} \tilde{v} : \widetilde{Q}_0 \setminus \tilde{\mathfrak{a}} &\hookrightarrow \widetilde{Q}_1 \\ \tilde{a}_{ij} &\mapsto (\tilde{v}_a)_{ij}. \end{aligned}$$

Statements (i)-(iii) follow immediately from the construction.

By Lemma 6.7.3 the arrow v_b is not involved in any differential but differentials of the distinguished loops $\partial(a_1)$ and $\partial(a_2)$, (if such distinguished loops exit). The matrix $F\big(v_b\big)$ contains an entry $\tilde{u}_{21} := F\big(v_b\big)_{21}$. Since the differential $\tilde{\partial}$ is "generated" by ∂, the dotted arrow $\tilde{u}_{21}$ is not involved in any differential $\tilde{\partial}(\tilde{x})$ for any $\tilde{x} \in \widetilde{Q}_0$, not even in the differential of distinguished loops $\tilde{\partial}(\tilde{a}_1)$ or $\tilde{\partial}(\tilde{a}_2)$. Lemma 6.6.2 implies the claim (iv). $\square$

Brick-reduction Algorithm

Let (Q, ∂) be a BT-differential biquiver with a minimal edge $b : 2 \to 1$, and $(\widetilde{Q}, \tilde{\partial})$ be the differential biquiver obtained from (Q, ∂) by the minimal edge reduction on b. Recall that the set of vertices of Q is $I = \{1, \ldots, n\}$ and the set of vertices of $\widetilde{Q}$ is $\tilde{I} = \{0, 1, 2, \ldots, n\}$. The corresponding Roiter bocses are denoted by $\mathscr{A}$ and $\widetilde{\mathscr{A}}$ respectively. Let M be a sincere brick of (Q, ∂) of vector dimension $(s_1, \ldots, s_n)$, with $s_1 \geq s_2$.

Combining the minimal edge reduction with restriction on $(\widetilde{Q}^1, \tilde{\partial})$ we obtain the composition of morphisms:

$$F : \mathscr{A} \to \mathsf{add}\, \widetilde{\mathscr{A}} \to \mathsf{add}\, \widetilde{\mathscr{A}}^{A_1},$$

where we take $A_1 = \widetilde{A}/(e_2)$ for $s_1 > s_2$ and $A_1 = \widetilde{A}/(e_1, e_2)$ if $s_1 = s_2$. (Here e_1, e_2 are idempotents of vertices 1 and 2 respectively). The induced fully faithful functor

$$F^* : \mathsf{Rep}(\widetilde{Q}^1, \tilde{\partial}) \hookrightarrow \mathsf{Rep}(\widetilde{Q}, \tilde{\partial}) \xrightarrow{\sim} \mathsf{Rep}(Q, \partial) \tag{6.27}$$

is dense on $\mathsf{Br}(Q, \partial)(\mathbb{s})$. The transfer from (Q, ∂) to $(\widetilde{Q}^1, \tilde{\partial})$ is called the *step of reduction* corresponding to M or to the nonempty stratum $\mathsf{Br}(Q, \partial)(\mathbb{s})$. There is a functorial bijection

$$\mathsf{Br}(Q, \partial)(s_1, \ldots, s_n) \cong \mathsf{Br}(\widetilde{Q}^1, \tilde{\partial})(s_2, s_1 - s_2, s_3, \ldots, s_n). \tag{6.28}$$

Indeed, if $\widetilde{M}$ is a brick of $(\widetilde{Q}^1, \tilde{\partial})$ such that $F^*(\widetilde{M}) = M$ then $\tilde{s}_i := \widetilde{M}(i) = M(i) = s_i$ for $i \in I \setminus \{1\}$. Moreover, $\widetilde{M}(0 \oplus 1) = M(1)$, and since $\widetilde{M}(2) = 0$, $\tilde{s}_0 := \widetilde{M}(0) = \widetilde{M}(2 \oplus 0) = M(2) = s_2$, thus $\tilde{s}_1 := \dim \widetilde{M}(1) = \dim(M)(1) - \dim M(2) = s_1 - s_2$.

Hence, the reduction for bricks of a given vector dimension $\mathbb{s}$ is equivalent to some kind of the Euclidean Algorithm.

The *Brick-reduction Algorithm* corresponding to the vector dimension $\mathbb{s}$ such that $\mathsf{Br}(Q, \partial)(\mathbb{s})$ is nonempty, is a sequence of m reduction steps:

$$\begin{aligned}
F_1 : \mathscr{A}_0 := \mathscr{A} &\longrightarrow \mathsf{add}\, \widetilde{\mathscr{A}} \longrightarrow \mathsf{add}(\mathscr{A}_1), \\
&\ \vdots \\
F_k : \mathscr{A}_{k-1} &\longrightarrow \mathsf{add}\, \widetilde{\mathscr{A}_{k-1}} \longrightarrow \mathsf{add}(\mathscr{A}_k), \\
&\ \vdots \\
F_m : \mathscr{A}_{m-1} &\longrightarrow \mathsf{add}\, \widetilde{\mathscr{A}_{m-1}} \longrightarrow \mathsf{add}(\mathscr{A}_m) = \mathsf{add}(R),
\end{aligned} \tag{6.29}$$

where after a proper reordering of vertices we can assume on each step $b : 2 \to 1$ is a minimal edge, $A_k := \widetilde{A_{k-1}}/(e_2)$ for $s_2 > s_1$, $A_k := \widetilde{A_{k-1}}/(e_1, e_2)$ for $s_1 = s_2$, $\mathscr{A}_k := \left(\widetilde{\mathscr{A}_{k-1}}\right)^{A_k}$, and R is either $\Bbbk$ or $\Bbbk[t]$. The composition of these morphisms gives rise to a functor

$$F := F_m \circ \ldots \circ F_1 : \mathscr{A} \longrightarrow \mathsf{add}(R). \tag{6.30}$$

Note that elements of the additive hull can be regarded as projective modules. Thus F can be considered as a representation of $\mathscr{A}$ over R, in accordance with the Definition B.0.5. Moreover, the induced functor $F^* : \mathsf{Rep}(R) \to \mathsf{Rep}(\mathscr{A})$ is fully faithful, and by the construction $F^*(N) = N \otimes_R F$ which implies that F is strict and brick-strict according to Definitions B.0.6 and B.0.8. Hence, there are two possible cases:

- if $R = \Bbbk$ then M is a unique brick in $\mathsf{Br}(Q, \partial)(\mathbb{s})$;
- if $R = \Bbbk[t]$ then bricks form one one-parameter family:

$$\mathsf{Br}(Q, \partial)(\mathbb{s}) = \mathbb{F}^s(F) = \{N(\lambda) \otimes_{\Bbbk[t]} F | N(\lambda) = \Bbbk[t]/(t - \lambda), \lambda \in \Bbbk\}.$$

Hence, the morphism F defines the canonical form of $\mathsf{Br}(Q, \partial)(\mathbb{s})$ over $\Bbbk[t]$ and the tensor product with $N(\lambda)$ corresponds to the substitution $t := \lambda$.

This is the reduction algorithm for a BT-differential biquiver $(Q, \tilde{\partial})$ and a non-empty stratum $\mathsf{Br}(Q, \tilde{\partial})(\mathbb{s})$. To make the statement rigorous we formulate it as a theorem.

Theorem 6.7.7. *I. A differential biquiver of* BT*-type is brick-tame.*

II. Let (Q, ∂) be a BT*-differential biquiver and $\mathbb{s} = (s_1, \dots, s_n)$ be a dimension vector such that the stratum* $\mathsf{Br}(Q, \partial)(\mathbb{s})$ *is non-empty. Then*

(i) $\mathsf{Br}(Q, \partial)(\mathbb{s})$ *consists either of a unique brick or of one one-parameter family of bricks;*

(ii) if (Q, ∂) is full then $\mathsf{Br}(Q, \partial)(\mathbb{s}) \cong \mathsf{Br}(\Bbbk[t])$; *and if F is the reduction functor as in (6.30) then for each $M \in$* $\mathsf{Br}(Q, \partial)(\mathbb{s})$ *there exists a brick $\lambda \in$* $\mathsf{Br}(R)$ *such that $M = F^{-1}(\lambda)$.*

Remark 6.7.8. According to B.0.8 the functor F can be considered as the tensor product $- \otimes_{\Bbbk[t]} K$, where K is a brick-strict representation of (Q, ∂) over $\Bbbk[t]$. In terms of matrices K is a *canonical form* of $\mathsf{Br}(Q, \partial)$. In other words, reversing algorithm 6.29 one can recover a canonical form of $M \in \mathsf{Br}(Q, \partial)(\mathbb{s})$.

Chapter 7

Applications of bocses technique

A differential biquiver (Q, ∂) such that $\mathsf{Rep}(Q, \partial) = \mathsf{BM}_P$ for one of the problems BM_P from Chapters 3,4 or 5 turns out to be of BT-type. Hence by Theorem 6.7.7 it is brick-tame. In the case of vector bundles (Q, ∂) is full and so, for a fixed vector dimension $\mathbb{s}$ there are either no bricks at all or they form one one-parameter family. In the case of torsion free sheaves, which are not vector bundles, for a fixed sincere vector dimension $\mathbb{s}$ if a brick exists then it is unique.

In this chapter we describe the course of brick-reduction for each problem BM_P. This enables us to answer the question when a brick of a given vector dimension exists. For vector bundles and torsion free sheaves on a cuspidal cubic curve we describe brick-reduction step-by-step. However, following this way in general on each step we should choose a minimal edge which is not unique. Thus the whole picture remains hidden. To improve this situation we provide an *automaton of brick reduction*, which is an oriented graph on the set of vertices called *states*, and whose arrows are possible *transitions* from a state to a state. Matrix problems appearing in course of brick-reduction are interpreted as states and possible steps of brick-reduction define the transitions.

For vector bundles on a cuspidal cubic curve the automaton is trivial. It consists of a unique state, since the problem is self-reproducing. The brick-reduction for vector bundles on Kodaira fiber III is described step-by-step and then encoded by an automaton. We also present the brick-reduction automaton for vector bundles on the Kodaira fiber IV. Then the brick-reduction for torsion free sheaves on the Kodaira fiber III can be encoded as a sub-automaton of it.

It is remarkable that only some special states of the brick-reduction automatons can be interpreted in terms of vector bundles or torsion free sheaves. Such states are called *principal.* In each case gluing paths we obtain a factor automaton with principal states only. A path p on it encodes a functorial bijection:

$$p : \mathsf{BM}^s_P(\mathbb{s}) \xrightarrow{\cong} \mathsf{BM}^s_P(\mathbb{s}'),$$

where $\mathbb{s} > \mathbb{s}'$ (i.e. $s_i \geq s'_i$ for all vertices i and there exists a vertex j such that $s_j > s'_j$). Coming back to the original classification problem we reformulate the result in terms of rank and multidegree of a vector bundle. It turns out that for a rank r and a multidegree $\mathbb{d}$ such that

$$g.c.d.(r, d) = 1, \tag{7.1}$$

where $d = \sum_i^N d_i$, there exists a path p such that

$$\begin{tikzcd}
\mathsf{VB}^s_E(r, \mathbb{d}) \arrow[d, "\cong"] & & \mathsf{Pic}^{(0,\ldots,0)}(E) \arrow[d, "\cong"] \\
\mathsf{BM}^s_P(\mathbb{s}) \arrow[rr, "p", "\sim"'] & & \mathsf{BM}^s_P(1, 0, \ldots, 0).
\end{tikzcd}$$

Reduction for torsion free sheaves with coprime rank and degree can be encoded analogously. We consider torsion free sheaves which are not vector bundles separately, since the relations between the multidegree and the degree in this case is different.

Short notations

For a BT-differential biquiver (Q, ∂) we introduce the following system of short notations:

- a vertex with a distinguished loop is denoted by a bullet $\bullet$;
- a vertex with no distinguished loop is denoted by a circle $\circ$;
- on this set of vertices we draw the graph with arrows $Q_0 \setminus \mathfrak{a}$;
- dotted arrows from $\mathrm{im}(v)$ are omitted; we visualize only those $u \in Q_1$ which do not belong to $\mathrm{im}(v)$;
- a differential of an arrow $x \in Q \setminus \mathfrak{a}$ is called *complete* if it contains all possible paths of degree $\deg(x) + 1$, for such arrows we omit the differentials.

Remark 7.0.1. The defined system of notations can be used for any BT-differential biquiver. However, we will see that in applications we deal only with biquivers such that $\mathrm{im}(v) = Q_1$ and each nondistinguished arrow is complete or minimal with respect to triangularity. In the course of reduction there appear some dotted arrows not from $\mathrm{im}(v)$ but these arrows are not involved in any differential and they guarantee that $\mathsf{Br}(Q, \partial)$ splits. After restricting (Q, ∂), the condition $\mathrm{im}(v) = Q_1$ holds again.

Example 7.0.2. The differential biquiver from Example 6.5.7

$$\begin{tikzcd}
\overset{1}{\bullet} \arrow[loop left, "a_1"] \arrow[r, dotted, bend right, "v"'] & \overset{2}{\bullet} \arrow[l, bend right, "b"']
\end{tikzcd}$$

with $\partial(b) = 0$ and $\partial(a_1) = -vb$ can be encoded by

$$P = \underset{1}{\bullet} \longleftarrow \underset{2}{\circ}$$

a step of brick-algorithm respectively is:

$$\underset{1}{\bullet} \longleftarrow \underset{2}{\circ} \implies \underset{1}{\bullet} \longleftarrow \underset{2}{\circ}$$

with the reduction of sizes: $(s_1, s_2) \to (s_1 - s_2, s_2)$.

7.1 Matrix problem for vector bundles on a cuspidal cubic curve

Let us illustrate the encoding system on the problem BM_P^s obtained for vector bundles on a cuspidal cubic curve. The poset

$$P = \overset{1}{\bullet} \longleftarrow \overset{2}{\bullet}$$

encodes the differential biquiver (Q_P, ∂_P) from Example 6.5.6:

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[loop left, "a_1"] \arrow[r, bend right, dotted, "v"'] & \overset{2}{\bullet} \arrow[l, bend right, "b"'] \arrow[loop right, "a_2"] \end{tikzcd}$$

and the differential:

$$\begin{aligned} \partial(b) &= 0 \\ \partial(a_1) &= -vb \\ \partial(a_2) &= bv. \end{aligned}$$

Assume there exists a sincere brick $M \in \mathsf{Br}(Q, \partial)(s_1, s_2)$ of vector dimension (s_1, s_2), (i.e. the stratum $\mathsf{Br}(Q, \partial)(s_1, s_2)$ is not empty). Then according to Example 6.6.3 we obtain

$$\mathsf{Br}(Q, \partial)(s_1, s_2) \cong \begin{cases} \mathsf{Br}(Q, \partial)(s_1 - s_2, s_2), & \text{if } s_1 > s_2, \\ \mathsf{Br}(Q, \partial)(s_1, s_2 - s_1), & \text{if } s_2 > s_1. \end{cases}$$

As a corollary of Theorem 6.7.7 we get the following rigorous statement:

Theorem 7.1.1. *The stratum $\mathsf{Br}(Q, \partial)(s_1, s_2)$ is not empty if and only if sizes s_1 and s_2 are coprime. In this case bricks $\mathsf{Br}(Q, \partial)(s_1, s_2)$ form a one one-parameter family. The canonical form of this family can be recovered by reversing the reduction algorithm.*

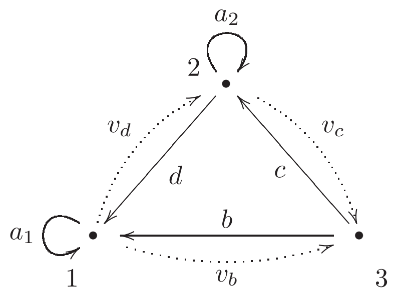
Thus we repeated the matrix considerations from Section 3 in terms of differential biquivers.

7.2 Matrix problem for torsion free sheaves on a cuspidal cubic curve

Let P be a poset encoding the matrix problem BM_P for simple torsion free sheaves on a cuspidal cubic curve:

$$P = \begin{tikzcd} & \bullet^2 \arrow[dl] & \\ \bullet^1 & & \circ^3 \arrow[ul] \arrow[ll] \end{tikzcd}.$$

The corresponding differential biquiver (Q_P, ∂_P) is



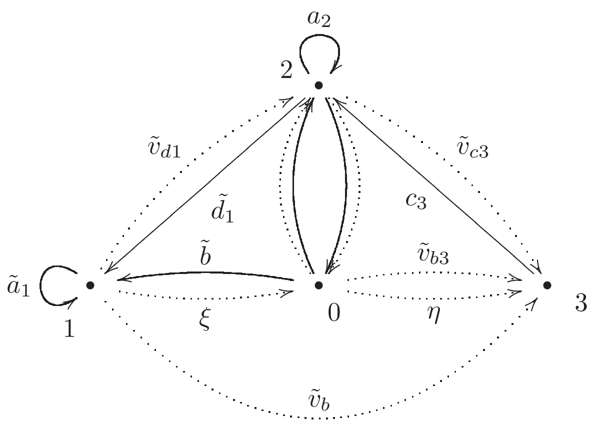
(7.2)

and differentials of all non distinguished arrows are complete:

$$\begin{aligned}
\partial(b) &= \partial(v_c) = \partial(v_d) = 0 \\
\partial(c) &= -v_d b \\
\partial(d) &= b v_c \\
\partial(a_1) &= b v_b + d v_d \\
\partial(a_2) &= c v_c - v_d d \\
\partial(v_b) &= v_c v_d.
\end{aligned}$$

Remark 7.2.1. Note that $(Q|_{\{1,2\}}, \partial)$ is a differential biquiver (6.23) i.e. the matrix problem for torsion free sheaves $\mathsf{Rep}(Q, \partial)$ contains the matrix problem for simple vector bundles as a subproblem.

Assume there exists a sincere brick $M \in \mathsf{Br}(Q, \partial)(s_1, s_2, s_3)$ of vector dimension $(s_1, s_2, s_3) \in \mathbb{N}^3$. After a minimal edge reduction and regularization of superfluous arrow $\tilde{a}_0$ we obtain the following differential biquiver $(\widetilde{Q}, \tilde{\partial})$:



Let $F(c) = (\tilde{c_0}, \tilde{c_3})$ and $F(d) = (\tilde{d_1}, \tilde{d_0})^T$. Since $F(\partial(c)) = \tilde{\partial}(F(c))$ and $F(\partial(d)) = \tilde{\partial}(F(d))$, we deduce the differential $\tilde{\partial}$:

$$\begin{aligned}
\tilde{\partial}(\tilde{c_0}) &= -v_{d0} + \tilde{c_3}\eta \\
\tilde{\partial}(\tilde{c_3}) &= 0 \\
\tilde{\partial}(\tilde{d_1}) &= 0 \\
\tilde{\partial}(\tilde{d_0}) &= -v_{c0} - \xi\tilde{d_1}.
\end{aligned}$$

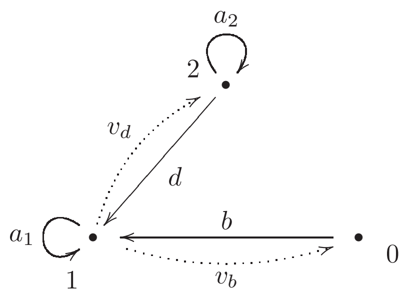
Moreover, arrows v_{c0} and v_{d0} are not involved in the differential of any other arrow. Thus arrows $\tilde{c_0} : 0 \to 2$, $\tilde{v}_{c0} : 0 \dashrightarrow 2$ $\tilde{d_0} : 2 \to 0$ and $\tilde{v}_{d0} : 2 \dashrightarrow 0$ can be deleted, using the regularization procedure. For simplicity we use the same notation $(\widetilde{Q}, \tilde{\partial})$. After a straightforward verification we obtain:

$$\begin{aligned}
\tilde{\partial}(\tilde{c_3}) &= \tilde{\partial}(\tilde{d_1}) = \tilde{\partial}(\tilde{b}) = 0 \\
\tilde{\partial}(\tilde{a_1}) &= d_1\tilde{v}_{d_1} + \tilde{b}\xi \\
\tilde{\partial}(\tilde{a_2}) &= -\tilde{v}_{d_1}d_1 + \tilde{c_3}\tilde{v}_{c3}.
\end{aligned}$$

By Proposition 6.7.6 taking into account that $s_1 \neq 0$, we obtain

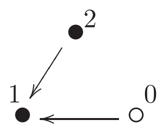
$$\mathsf{Br}(Q, \partial)(s_1, s_2, s_3) \cong \mathsf{Br}(\widetilde{Q}|_{\{0,1,2\}}, \tilde{\partial})(s_1 - s_3, s_2, s_3)$$

Omitting indices, since it does not lead to confusions we get $\widetilde{Q}|_{\{0,1,2\}}$:



(7.3)

where a_1, a_2 are distinguished loops and $\tilde{\partial}(b) = \tilde{\partial}(d) = 0$. This differential biquiver is encoded by the poset:

$P =$ 

and both arrows are minimal. It seems reasonable to rename the vertex "0" by "3." Analogously, applying minimal edge reduction on $3 \to 1$ to $(\widetilde{Q}|_{\{0,1,2\}}, \tilde{\partial})$

obtain

$$\begin{tikzcd}[column sep=small]
& \bullet^{2} \arrow[dl] & & & & \bullet^{2} \arrow[dl] & & & & \bullet^{2} \arrow[dl] \arrow[dr] & \\
\bullet^{1} & & \circ^{3} \arrow[ll] \arrow[ul] & \Longrightarrow & \bullet^{1} & & \circ^{3} \arrow[ll] & \Longrightarrow & \bullet^{1} & & \circ^{3} \arrow[ll]
\end{tikzcd} \tag{7.4}$$

and reduction of sizes:

$$(s_1, s_2, s_3) \to (s_1 - s_3, s_2, s_3) \to (s_1 - 2s_3, s_2, s_3).$$

Obtained problem $\mathsf{Br}(Q, \partial)$ is self-reproducing in some sence. Indeed, if $s_1 > s_2$ the reductions along minimal arrows $2 \to 1$ and $3 \to 1$ is:

$$\begin{tikzcd}[column sep=small]
& \bullet^{2} \arrow[dl] \arrow[dr] & & & & \bullet^{2} \arrow[dl] & & & & \bullet^{2} \arrow[dl] \arrow[dr] & \\
\bullet^{1} & & \circ^{3} \arrow[ll] & \Longrightarrow & \bullet^{1} & & \circ^{3} \arrow[ll] & \Longrightarrow & \bullet^{1} & & \circ^{3} \arrow[ll]
\end{tikzcd} \tag{7.5}$$

with reduction for sizes:

$$(s_1, s_2, s_3) \to (s_1 - s_2, s_2, s_3) \to (s_1 - s_2 - s_3, s_2, s_3)$$

or if $s_1 < s_2$ we proceed along $2 \to 1$ and $2 \to 3$:

$$\begin{tikzcd}[column sep=small]
& \bullet^{2} \arrow[dl] \arrow[dr] & & & & \bullet^{2} \arrow[dl] \arrow[dr] & & & & \bullet^{2} \arrow[dl] \arrow[dr] & \\
\bullet^{1} & & \circ^{3} \arrow[ll] & \Longrightarrow & \bullet^{1} & & \circ^{3} & \Longrightarrow & \bullet^{1} & & \circ^{3} \arrow[ll]
\end{tikzcd} \tag{7.6}$$

$$(s_1, s_2, s_3) \to (s_1, s_2 - s_1, s_3,) \to (s_1, s_2 - s_1 - s_3, s_3)$$

On each step of reduction we choose a minimal edge as follows: if the vertex 3 with no distinguished loop is attached to a minimal edge then we reduce it first. This rule determines a minimal edge on each step of reduction. For any minimal edge $i \to 3$ or $3 \to i$ we have $s_3 < s_i$. Since otherwise, taking $S_1 = S_2 = \mathrm{id}$, $S(v_b) = (0, A)^T$, where A is a nonzero matrix, we construct a nontrivial endomorphism.

Note that if $s_1 = s_2$ on the first step of reduction (7.5) then there are no sincere bricks at all. The category $\mathsf{Br}(\widetilde{Q}|_{1,3}, \tilde{\partial})$ splits:

$$\begin{tikzcd}[column sep=small]
& \bullet^{2} \arrow[dl] \arrow[dr] & & & & \\
\bullet^{1} & & \circ^{3} \arrow[ll] & \Longrightarrow & \bullet^{1} & \circ^{3}\,.
\end{tikzcd} \tag{7.7}$$

Hence, the stratum (s_1, s_2, s_3) is not empty only if on each step we have reduction as in (7.5) or (7.6), with the final step either if $s_1 = s_2 + s_3$

$$\begin{tikzcd}[column sep=small]
& \bullet^{2} \arrow[dl] \arrow[dr] & & & & \bullet^{2} \arrow[dl] & & & & \\
\bullet^{1} & & \circ^{3} \arrow[ll] & \Longrightarrow & \bullet^{1} & & \circ^{3} \arrow[ll] & \Longrightarrow & {}^{2}\bullet & \circ^{3} \arrow[l]
\end{tikzcd} \tag{7.8}$$

and $(s_1, s_2, s_3) \to (s_1 - s_2, s_2, s_3) \to (s_2, s_3)$; or if $s_2 = s_1 + s_3$

$$
\begin{tikzcd}[column sep=small]
 & \bullet^{2} \arrow[dl] \arrow[dr] & \\
\underset{1}{\bullet} & & \underset{3}{\circ} \arrow[ll]
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}[column sep=small]
 & \bullet^{2} \arrow[dl] \arrow[dr] & \\
\underset{1}{\bullet} & & \underset{3}{\circ}
\end{tikzcd}
\Longrightarrow
{}^{1}\bullet \longrightarrow \circ^{3}
\tag{7.9}
$$

with $(s_1, s_2, s_3) \to (s_1, s_2 - s_1, s_3) \to (s_1, s_3)$.
Obtained problem is the problem from Example 7.2.4. Hence, we can draw the conclusion $s_3 = 1$, and applying the reduction to the original problem (7.2) we get:

Lemma 7.2.2. *The sincere stratum* $\mathsf{Br}(Q, \partial)(s_1, s_2, s_3)$ *of the problem*

$$
\begin{tikzcd}[column sep=small]
 & \bullet^{2} \arrow[dl] \arrow[dr] & \\
\underset{1}{\bullet} & & \underset{3}{\circ} \arrow[ll]
\end{tikzcd}
$$

is non-empty if and only if $s_3 = 1$ *and* $(s_1 + 1)$ *is coprime with* $(s_2 + 1)$.

Reformulating it for the original problem (7.2) we obtain

Lemma 7.2.3. *The sincere stratum* $\mathsf{Br}(Q, \partial)(s_1, s_2, s_3)$ *of the problem*

$$
\begin{tikzcd}[column sep=small]
 & \bullet^{2} \arrow[dl] & \\
\underset{1}{\bullet} & & \underset{3}{\circ} \arrow[ll] \arrow[ul]
\end{tikzcd}
.
$$

is non-empty only if $s_3 = 1$ *and* $(s_1 - 1)$ *is coprime with* $(s_2 + 1)$.

Canonical forms

Let us illustrate the reduction on an matrices to see how the canonical form can be recovered from the reduction algorithm.

Example 7.2.4. Let us consider reduction from the Example 7.0.2:

$$
{}^{1}\bullet \longleftarrow \circ^{2} \Longrightarrow {}^{1}\bullet \longleftarrow \circ^{2}
$$

This problem is self-reproducing with a unique type of reduction. For any dimension $(n, 1)$ we can write a canonical form B_n as follows $B_1 = \begin{array}{cc|c} 1 & 2 & \\ \boxed{0} & \boxed{1} & 1 \end{array}$

and a step of induction:

$$B_n = \begin{array}{cc} \begin{array}{cc} \qquad 1 \qquad & \quad 2 \end{array} & \\ \left[\begin{array}{c|c} \qquad B_{n-1} \qquad & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 0 \quad \dots \quad 0 & 1 \end{array}\right] & 1 \end{array}$$

Finally for sincere bricks we get: the stratum $\mathsf{Br}(Q,\partial)(s_1,s_2)$ is empty if $s_2 \geq 1$ and stratum $\mathsf{Br}(Q,\partial)(n,1)$ consists of a unique brick with the canonical form:

$$B_n = \begin{array}{cc} \begin{array}{cc} \qquad 1 \qquad & \quad 2 \end{array} & \\ \left[\begin{array}{cccc|c} 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{array}\right] & 1 \end{array} \tag{7.10}$$

Note that the canonical form for the "dual" differential biquiver $\overset{1}{\bullet} \longrightarrow \overset{2}{\circ}$ is the transposed matrix B_n^t.

Example 7.2.5. Let us describe a canonical form of a brick $B \in \mathsf{Br}(Q,\partial)$ of of the differential biquiver (Q,∂) :

$$\begin{tikzcd} & \overset{2}{\bullet} \arrow[dl] \arrow[dr] & \\ \overset{1}{\bullet} & & \overset{3}{\circ} \arrow[ll] \end{tikzcd}$$

We construct the canonical form inductively by reversing reductions (7.8), (7.9) (7.5) and (7.6). As usual, empty blocks denote nonexisting blocks. We stress the difference between them and zero blocks, to keep the correspondence between matrices and representations of differential biquivers.

Let B be a brick of vector dimension $(n,1,n+1)$, using short notations for the blocks of the canonical form from Example 7.0.2: $(X,Y)^t := B_n$, we obtain the following canonical form:

$$\begin{array}{c|c|c|} & 3 & 2 \\ \hline 3 & & Y \\ \hline 2 & & X \\ \hline \end{array} \Longrightarrow \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 1 & 0 & Y & 1 \\ \hline 2 & & X & \\ \hline 3 & & & \\ \hline \end{array} \Longrightarrow \begin{array}{c|c|c|c|c|} & 1 & 3 & \multicolumn{2}{c|}{2} \\ \hline \multirow{2}{*}{1} & 0 & Y & 1 & 0 \\ & 0 & X & 0 & \mathbb{I}_n \\ \hline 3 & \multicolumn{2}{c|}{} & & 0 \\ \hline 2 & \multicolumn{2}{c|}{} & & 0 \\ \hline \end{array}. \tag{7.11}$$

Let B be a brick of vector dimension $(n+1, 1, n)$. Induction of the canonical forms corresponding to the reduction (7.8) is the same as above but with transposed matrices, thus using the short notations $(X, Y) := B_n$ we obtain:

$$
\begin{array}{c|c|c|}
 & 1 & 3 \\ \hline
1 & X & Y \\ \hline
3 & & \\ \hline
\end{array}
\Longrightarrow
\begin{array}{c|c|c|c|}
 & 3 & 1 & 2 \\ \hline
3 & & & 1 \\ \hline
1 & & X & Y \\ \hline
2 & & & 0 \\ \hline
\end{array}
\Longrightarrow
\begin{array}{c|c|c|cc|}
 & 1 & 3 & 2 & \\ \hline
1 & 0 & 0 & \mathbb{I}_n & 0 \\ \hline
3 & & & 0 & 1 \\ \hline
2 & & & X & Y \\
 & & & 0 & 0 \\ \hline
\end{array}. \qquad (7.12)
$$

Analogously, we can rewrite reduction (7.5) in terms of matrices: Assume that $B \in \mathsf{Br}(Q, \partial)(s_1, s_2, s_3)$ is given, and consists of the following blocks

$$
B = \begin{array}{c|c|c|c|}
 & 1 & 3 & 2 \\ \hline
1 & X & Y & Z \\ \hline
3 & & & W \\ \hline
2 & & & T \\ \hline
\end{array}
$$

then $B \in \mathsf{Br}(Q, \partial)(s_1, s_2 + s_1 + s_3, s_3)$ we get

$$
\begin{array}{c|c|c|c|}
 & 1 & 3 & 2 \\ \hline
1 & X & Y & Z \\ \hline
3 & & & W \\ \hline
2 & & & T \\ \hline
\end{array}
\Longrightarrow
\begin{array}{c|c|c|cc|}
 & 3 & 1 & 2 & \\ \hline
3 & & 0 & 1 & 0 \\ \hline
1 & & X & Y & Z \\ \hline
2 & & & 0 & W \\
 & & & 0 & T \\ \hline
\end{array}
\Longrightarrow
\begin{array}{c|c|c|ccc|}
 & 1 & 3 & 2 & & \\ \hline
1 & 0 & 0 & \mathbb{I}_{s_1} & 0 & 0 \\ \hline
3 & & & 0 & 1 & 0 \\ \hline
2 & & & X & Y & Z \\
 & & & 0 & 0 & W \\
 & & & 0 & 0 & T \\ \hline
\end{array} \qquad (7.13)
$$

analogously for $B \in \mathsf{Br}(Q, \partial)(s_1 + s_2 + s_3, s_2, s_3)$ we get

$$
\begin{array}{c|c|c|c|}
 & 1 & 3 & 2 \\ \hline
1 & X & Y & Z \\ \hline
3 & & & W \\ \hline
2 & & & T \\ \hline
\end{array}
\Longrightarrow
\begin{array}{c|cc|c|c|}
 & 1 & & 2 & 3 \\ \hline
1 & X & Y & Z & 0 \\
 & 0 & 0 & W & 1 \\ \hline
2 & & & T & \\ \hline
3 & & & & \\ \hline
\end{array}
\Longrightarrow
\begin{array}{c|ccc|c|c|}
 & 1 & & & 2 & 3 \\ \hline
1 & X & Y & Z & 0 & 0 \\
 & 0 & 0 & W & 1 & 0 \\
 & 0 & 0 & T & 0 & \mathbb{I}_{s_2} \\ \hline
2 & & & & 0 & 0 \\ \hline
3 & & & & & \\ \hline
\end{array} \qquad (7.14)
$$

Let us illustrate reduction (7.4) on matrices either:

$$
\begin{array}{c|c|c|c|}
 & 1 & 3 & 2 \\ \hline
1 & X & Y & Z \\ \hline
3 & & & W \\ \hline
2 & & & T \\ \hline
\end{array}
\Longrightarrow
\begin{array}{c|cc|c|c|}
 & 1 & & 2 & 3 \\ \hline
1 & X & Y & Z & 0 \\
 & 0 & 0 & W & 1 \\ \hline
2 & & & T & \\ \hline
3 & & & & \\ \hline
\end{array}
\Longrightarrow
\begin{array}{c|ccc|c|c|}
 & 1 & & & 2 & 3 \\ \hline
1 & X & Y & 0 & Z & 0 \\
 & 0 & 0 & 1 & W & 0 \\
 & 0 & 0 & 0 & 0 & 1 \\ \hline
2 & & & & T & 0 \\ \hline
3 & & & & & \\ \hline
\end{array}
\tag{7.15}
$$

7.3 Matrix problem for vector bundles on a tacnode curve

Using the system of short notations from the previous section the matrix category BM_P from Section 4 correspond to the category of representations of the differential biquiver

$$
\begin{tikzcd} & \bullet^2 \arrow[dl] & \\ \bullet^1 & & \bullet^3 \arrow[ll] \end{tikzcd}.
\quad \text{or its dual} \quad
\begin{tikzcd} & \bullet^2 & \\ \bullet^1 & & \bullet^3 \arrow[ll] \arrow[ul] \end{tikzcd}
\tag{7.16}
$$

Without loss of generality let us give the minimal edge reduction on edge $b : 3 \to 1$:

$$
\underset{(s_1, s_2, s_3)}{\begin{tikzcd} & \bullet^2 \arrow[dl] & \\ \bullet^1 & & \bullet^3 \arrow[ll] \end{tikzcd}}
\Longrightarrow \text{ either }
\underset{(s_1, s_2, s_3 - s_1)}{\begin{tikzcd} & \bullet^2 \arrow[dl] & \\ \bullet^1 & & \bullet^3 \arrow[ll] \end{tikzcd}}
\text{ or }
\underset{(s_1 - s_3, s_2, s_3)}{\begin{tikzcd} & \bullet^2 \arrow[dl] \arrow[dr] & \\ \bullet^1 & & \bullet^3 \arrow[ll] \end{tikzcd}}
\tag{7.17}
$$

$$
\underset{(s_1, s_2, s_3)}{\begin{tikzcd} & \bullet^2 & \\ \bullet^1 & & \bullet^3 \arrow[ll] \arrow[ul] \end{tikzcd}}
\Longrightarrow \text{ either }
\underset{(s_1 - s_3, s_2, s_3)}{\begin{tikzcd} & \bullet^2 & \\ \bullet^1 & & \bullet^3 \arrow[ll] \arrow[ul] \end{tikzcd}}
\text{ or }
\underset{(s_1, s_2, s_3 - s_1)}{\begin{tikzcd} & \bullet^2 & \\ \bullet^1 \arrow[ur] & & \bullet^3 \arrow[ll] \arrow[ul] \end{tikzcd}}
\tag{7.18}
$$

The new configuration in its turn can be reduced as follows:

$$
\underset{(s_1, s_2, s_3)}{\begin{tikzcd} & \bullet^2 \arrow[dl] & \\ \bullet^1 & & \bullet^3 \arrow[ll] \arrow[ul] \end{tikzcd}}
\Longrightarrow \text{ either }
\underset{(s_1 - s_3, s_2, s_3)}{\begin{tikzcd} & \bullet^2 & \\ \bullet^1 & & \bullet^3 \arrow[ll] \arrow[ul] \end{tikzcd}}
\text{ or }
\underset{(s_1, s_2, s_3 - s_1)}{\begin{tikzcd} & \bullet^2 \arrow[dl] & \\ \bullet^1 & & \bullet^3 \arrow[ll] \end{tikzcd}}
\tag{7.19}
$$

we do not prove this reduction, since the reduction is completely analogous to the reduction of non full biquiver (7.2), which was considered explicitly into the previous section.

Analogously to the previous section there are following possibilities for the final step of brick-reduction.

Final steps. Note that the following configuration splits:

$$\begin{tikzcd} & \bullet^{2} \arrow[dl] & & & \\ \bullet^{1} & & \bullet^{3} \arrow[ul] \arrow[ll] & \Longrightarrow & \bullet^{2} \quad \bullet^{3} \end{tikzcd} \tag{7.20}$$

and if $s_1 = s_3$

then there is no sincere bricks. The other cases: if $s_1 = s_3$

$$\begin{tikzcd} & \bullet^{2} \arrow[dl] & & & \\ \bullet^{1} & & \bullet^{3} \arrow[ll] & \Longrightarrow & \bullet^{1} & \bullet^{2} \arrow[l] \end{tikzcd} \tag{7.21}$$

and $(s_1, s_2, s_3) \to (s_2, s_3)$; or if $s_2 = s_3$ then

$$\begin{tikzcd} & \bullet^{2} & & & \\ \bullet^{1} & & \bullet^{3} \arrow[ul] \arrow[ll] & \Longrightarrow & \bullet^{1} & \bullet^{3} \arrow[l] \end{tikzcd} \tag{7.22}$$

and $(s_1, s_2, s_3) \to (s_2, s_3)$. Obtained problem is the problem from Example 6.5.6 treated either in Section 7.1.

Remark 7.3.1. The course of the brick-reduction depends on the choice of a minimal edge. We can formulate a rule how to choose this edge: We start with reduction on the minimal arrow connecting vertices 1 and 3. Such minimal arrow exists for both configurations of (7.16). Apply the brick-reduction on this edge as long as possible. Then we uniquely obtain the next minimal edge. This new minimal edge we also reduce as long as possible and get the next one, etc. Following this way the complete picture of possible reductions remains hidden. To improve this situation we introduce the automaton of the brick-reduction.

7.4 Brick-reduction automaton

Definition 7.4.1. By a *brick-reduction automaton* we understand an oriented graph on the set of *internal states* Γ and with the set of arrows X, where

- Γ is a finite set of differential biquivers on the same finite set of vertices I;

- $X = \bigcup\limits_{\gamma\in\Gamma} X_\gamma$, where $X_\gamma = \{m \in Q_0(\gamma) \mid \partial(m) = 0\} \times \mathbb{Z}_2$ is a set of minimal solid arrows of γ with alternative orientations. In other words, $X_\gamma \times \mathbb{Z}_2$ is the set of possible steps of minimal edge reductions $x : \gamma \mapsto \gamma'$ from the state γ;

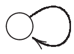
- An arrow $x \in X$ acts on the space of sizes $\mathbb{N}^{|I|}$ as follows: for a fixed state $\gamma \in \Gamma$ and a fixed tuple of sizes $\mathbb{s} \in \mathbb{N}^{|I|}$ we have:
  - if $s_i \geq s_j$ then the reduction $x = (i \to j, +) : \gamma \to \gamma' \in X_\gamma$ reduces $\mathbb{s}$ by the rule $s_i \mapsto s_i - s_j$ and $s_k \mapsto s_k$ for $k \in I \setminus \{i\}$, if $s_i > s_j$ then x can not be applied to the tuple $\mathbb{s}$;
  - analogously if $s_i \leq s_j$ then the reduction $y = (i \to j, -) : \gamma \to \gamma'' \in X_\gamma$ reduces $\mathbb{s}$ by the rule $s_j \mapsto s_j - s_i$ and $s_k \mapsto s_k$ for $k \in I \setminus \{j\}$, otherwise if $s_i > s_j$ then y can not be applied to $\mathbb{s}$.

A state γ and a tuple of sizes $\mathbb{s}$ encode an iso-class of canonical forms. We construct a canonical form (i.e. carry out a brick-reduction (6.29)) along a path on the automaton.

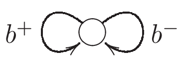
Definition 7.4.2.
- A sequence $p := (x_1x_2\dots x_n)$, $x_i \in X$ is called a *path* if the target of x_i coincides with the source of x_{i+1}. Clearly, a path operates on the set of sizes: $p : \mathbb{s} \mapsto \mathbb{s}'$, where $\mathbb{s} \geq \mathbb{s}'$ i.e. $s_i \geq s_i'$ for all $i \in I$.
- Two paths p_1 and p_2 with the a common source and a common target are called *equivalent* if for any tuple of sizes $(\mathbb{s}) \in \mathbb{N}^3$ $p_1(\mathbb{s}) = p_2(\mathbb{s})$.
- The semigroup of paths modulo the equivalence relation is called the *semigroup of the automaton* and denoted by Ω.

Remark 7.4.3. Note that if some sizes are zero then the reduction degenerates (see for example, (7.8) or (7.9)) but it still can be encoded by paths on the automaton. The transition (minimal edge reduction) on arrows attached to a vertex with zero size does not change sizes (i.e. "empty action"). We can proceed until the tuple of sizes $\mathbb{s}$ becomes $(0, \dots, 0, \overset{i}{1}, 0 \dots, 0)$ for some place i. However, we can always choose a path p such that $p(\mathbb{s}) = (1, 0, \dots, 0)$.

Example 7.4.4. The brick-reduction from Example 6.5.7 can be encoded as

 .

Analogously the brick-reduction from Section 7.1 (see also Example 6.5.6) can be encoded by the automaton (X, Γ) :

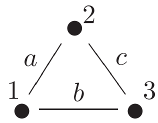


, (7.23)

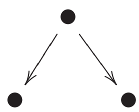
where $X = \mathbb{Z}_2 = \{b^+, b^-\}$ and $\Gamma = \{\gamma\}$ consists of a unique state.

7.5 Automaton for simple vector bundles on Kodaira fiber III

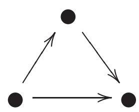
Let us encode the course of reduction from the previous section. For this purpose we fix vertices and give names to edges of a triangle:



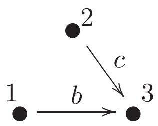
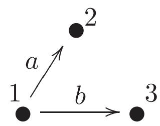
An arrow is encoded by the underlying edge and its orientation. Let us fix notations: $(a)^+ : 1 \to 2$ this means $(a)^- : 2 \to 1$; $(b)^+ : 1 \to 3$ and $(c)^+ : 2 \to 3$. In the course of brick-reduction we deal with posets of one of the forms:



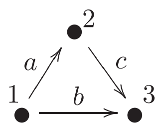
and



where the first configuration have the dual form and the second one is selfdual. A configuration of the first type is called *positiv oriented* if a minimal vertex is unique. Consequently, the dualform is called *negative oriented*. We denote the set such posets and their duals by Γ_1. A poset (as well as a differential biquiver) of this type can be encoded by a vertex and an orientation. Sometimes it is useful to encode this vertex by the attached edges. For example:

$(3)^+ = (bc)^+ =$  , $(1)^- = (ab)^- =$  .

A configuration of the second type can be encoded by its minimal[1] arrow, or equivalently, by its minimal and maximal vertices. For example:

$(b)^+ = (31) =$  .

The set of posets of the second type is denoted by Γ_2. Let P be a fixed poset of the first or the second type. Then an arrow can be encoded by the underlying edge, since the orientation of it is uniquely determined by the configuration. Let m be an edge of P connecting the maximal vertex i and the minimal vertex j. For $s_i \geq s_j$ a step of reduction is denoted by m^+ and respectively for $s_i \leq s_j$ by m^-. If $s_i = s_j$ both reductions coincide.

[1]with respect to triangularity

Automaton

The brick-reduction from the previous section can be encoded as follows:

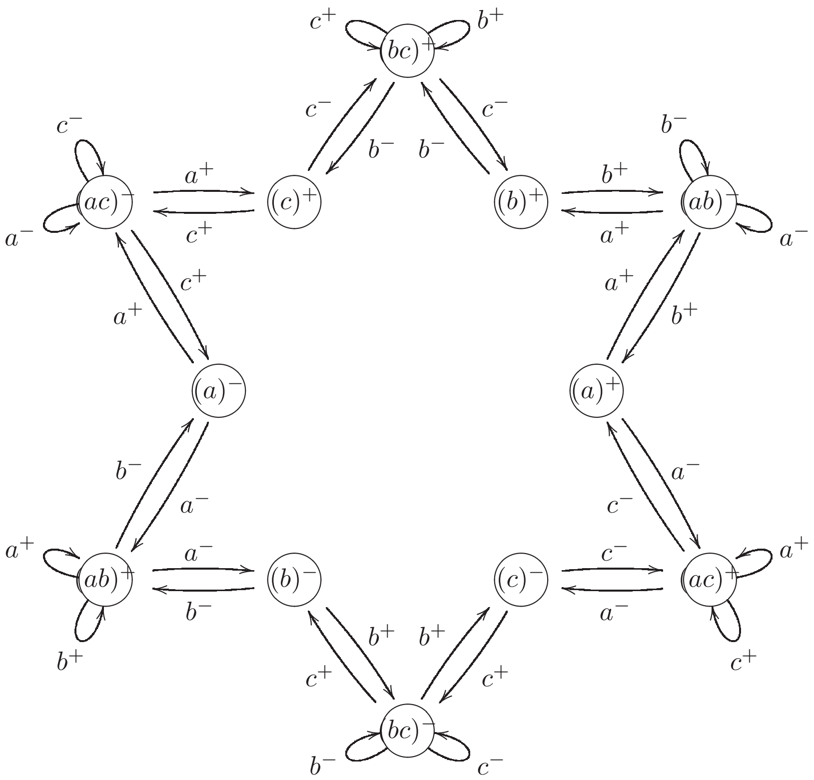


Figure 7.5.1 Brick-reduction automaton.

Alphabet X operates on the set of sizes: for a state $\gamma \in \Gamma$ and its minimal arrow $(m)^\sigma : i \to j$ the edge reduction acts as follows: $m^+((s_i, s_j, s_k)) = (s_i - s_j, s_j, s_k)$; and $m^-((s_i, s_j, s_k)) = (s_i, s_j - s_i, s_k)$; i.e.

$$m^+ \text{acts on sizes} (s_i, s_j, s_k) \text{ as } \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } m^- \text{ acts as } \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus we have described the action of X on the space of sizes $\mathbb{N}^3$.
We have the following obvious lemma:

Lemma 7.5.1. *The equivalence on the brick-reduction automaton 7.5.1 is determined by commutation relations*

$$m^\sigma n^\sigma \sim n^\sigma m^\sigma$$

for $m, n \in \{a, b, c\}$ *and* $\sigma \in \mathbb{Z}_2$.

The reduction stops at a state of Γ_1, since otherwise the representation splits. States of Γ_1 are called *principal*. Consider the automaton on principal states only.

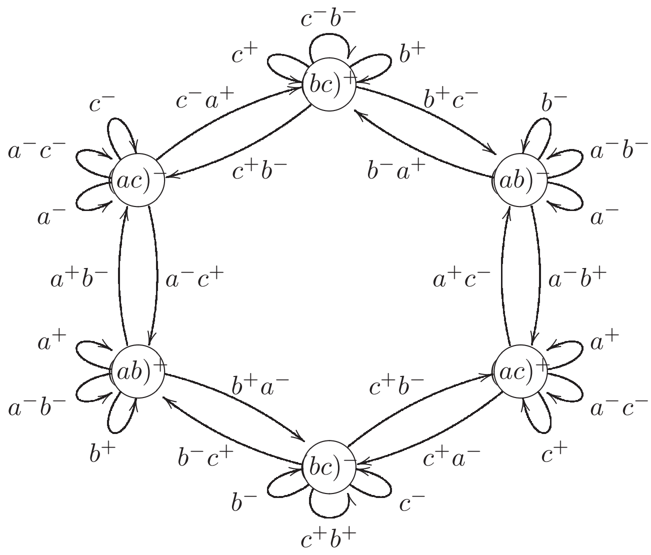


Figure 7.5.2 Principal reduction automaton.

Such a reduction can be explained in terms of bundles, since contrary to the automaton 7.5.1 this time each state encodes a matrix problem describing simple vector bundles. Thus each path p on the principal reduction automaton corresponds to a reduction

$$\mathsf{VB}^s_E(r, \mathbb{d}) \xrightarrow{p} \mathsf{VB}^s_E(r', \mathbb{d}'),$$

where $r' < r$.

Theorem 7.5.2. *Let $(i)^\sigma = (mn)^\sigma$ and $(i')^{\sigma'} = (m'n')^{\sigma'}$ be two states of Γ_1 connected by a path $p : (mn)^\sigma \to (m'n')^{\sigma'}$. Then*

$$g.c.d(s_j + s_k, s_i) = g.c.d(s'_{j'} + s'_{k'}, s'_{i'}),$$

where $j, k \in \{1, 2, 3\} \setminus \{i\}$ and $j', k' \in \{1, 2, 3\} \setminus \{i'\}$.

Proof. It is sufficient to prove the statement on arrows:

$$\begin{aligned} b^+,\ c^+,\ b^-c^- &: (bc)^+ \longrightarrow (bc)^+ \\ b^+c^- &: (bc)^+ \longrightarrow (ab)^- \\ c^+b^- &: (bc)^+ \longrightarrow (ac)^-. \end{aligned}$$

Indeed,

- $b^+ : (s_1, s_2, s_3) \mapsto (s_1 - s_3, s_2, s_3)$ and hence $(s_1 + s_2, s_3) \mapsto (s_1 + s_2 - s_3, s_3)$;
- $c^+ : (s_1, s_2, s_3) \mapsto (s_1, s_2 - s_3, s_3)$ and $(s_1 + s_2, s_3) \mapsto (s_1 + s_2 - s_3, s_3)$;
- $b^-c^- : (s_1, s_2, s_3) \mapsto (s_1, s_2, s_3 - (s_1 + s_2))$ and $(s_1 + s_2, s_3) \mapsto (s_1 + s_2, s_3 - (s_1 + s_2))$;

- b^+c^- : $(s_1, s_2, s_3) \mapsto (s_1 + s_2 - s_3, s_2, s_3 - s_2)$ and $(s_1 + s_2, s_3) \mapsto (s_3, s_1 + s_2 - s_3)$;

- c^+b^- : $(s_1, s_2, s_3) \mapsto (s_1, s_1 + s_2 - s_3, s_3 - s_1)$ and $(s_1 + s_2, s_3) \mapsto (s_3, s_1 + s_2 - s_3)$.

$\square$

Corollary 7.5.3. *Let $(s_1, s_2, s_3) \in \mathbb{N}^3$ be a triple such that*

$$g.c.d.(s_i, s_j + s_k) = 1 \tag{7.24}$$

where i, j, k are different indices. Then there exists a path p on the principal matrix reduction automaton starting at the state $(i)^\sigma = (mn)^\sigma$ such that $p(s_1, s_2, s_3) = (1, 0, 0)$.

Construction of a canonical form

Let Ω^* be a semigroup generated by paths p^* obtained form $p \in \Omega$ by reversing arrows. If a path p encodes a matrix reduction, then the reversed path p^* encodes the inductive construction of the canonical form.

Assume we have an input state $(i)^\sigma = (mn)^\sigma$ and a tuple of sizes (s_1, s_2, s_3) satisfying (7.24). To find a canonical form K we proceed as follows:

- find a path p such that $p(s_1, s_2, s_3) = (1, 0, 0)$;

- on the state $p(\gamma)$ input $\lambda \in \Bbbk$ which is a canonical form of sizes $(1, 0, 0)$;

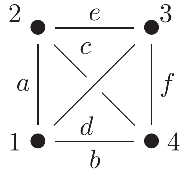
- Construct the canonical form K of sizes (s_1, s_2, s_3) inductively along the path p^* (i.e. $K = p^*(\lambda)$).

In Example 4.4.2 we construct a canonical form of a vector bundle of rank 9 and multidegree (3,4).

7.6 Automaton for simple vector bundles on Kodaira fiber IV

In this section we consider a full differential biquiver on four vertices obtained as a reformulation of the classification of vector bundles on the Kodaira fiber IV. The matrix problem for torsion free sheaves on a tacnode curve we postpone to the next section.

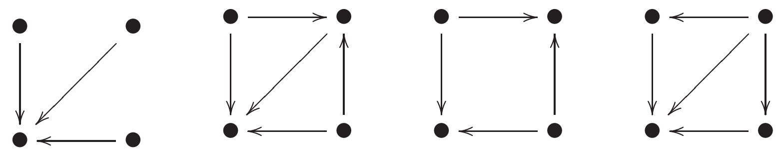
Following the similar lines to those of the previous section we give names to edges of a graph on four vertices:



Then an arrow is defined by the underlying edge and its orientation. We fix orientations as follows: $(a)^+ : 1 \to 2$ (that means $(a)^- : 2 \to 1$), $(b)^+ : 1 \to 4$, $(c)^+ : 2 \to 4$, $(d)^+ : 3 \to 1$, $(e)^+ : 3 \to 2$ and $(f)^+ : 3 \to 4$.

Differential biquivers obtained in the course of the brick-reduction

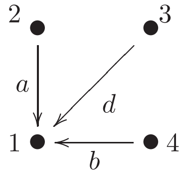
In the course of the brick-reduction we obtain differential biquiver, whose solid quiver Q_0 is determined by a poset P of one of the following forms:



A B C D

where A and B have dual posets and C and D are selfdual. The full differential biquivers on diagrams $A - D$ and dual ones, which we obtain in course of brick-reduction, can be described as follows:

- To define a differential biquiver on the diagram A or its dual it is sufficient to choose a vertex and its orientation (i.e. to say is it minimal or maximal). A differential biquiver on a poset A is denoted by $A^\sigma(i)$, where $i \in I$, $\sigma \in \{+, -\}$ and "+" indicates that the vertex i is minimal, respectively "–" asserts that i is maximal. For example:

  $A^+(1) =$ 

  Clearly, non-distinguished arrows in the corresponding differential biquiver are minimal. Altogether there are 8 differential biquivers of type A.

- To define a differential biquiver on the diagram B or its dual it is sufficient to choose the *diagonal* of the configuration and its orientation. A differential biquiver is encoded as $B^\sigma(i, j)$, where $i, j \in I$ determine the diagonal and $\sigma \in \{+,-\}$ is its orientation. For example:

$$B^{+}(1,3) = \begin{tikzcd} {}^{2}\bullet \arrow[r, "e"] \arrow[d, "a"'] & \bullet^{3} \arrow[dl, "d"] \\ {}_{1}\bullet & \bullet_{4} \arrow[l, "b"] \arrow[u, "f"'] \end{tikzcd}$$

Arrows a and b are minimal. Differentials of all other arrows are determined by completeness. Altogether there are 24 differential biquivers of type B.

- A differential biquiver on the diagram C is not linear. (Hence, the category of its representations can not be interpreted as BM_P. It is not a bimodule problem.) The biquiver is denoted by $C(i,j)$, for $i,j \in I$ and $C(i,j) = C(j,i)$. For example:

$$C(1,3) = \begin{tikzcd} {}^{2}\bullet \arrow[r, "e"] \arrow[d, "a"'] & \bullet^{3} \\ {}_{1}\bullet & \bullet_{4} \arrow[l, "b"] \arrow[u, "f"'] \end{tikzcd}$$

There are two ways to define a differential on the biquiver of type C by the completeness rule. For this purpose we should fix a minimal edge. But for convenience we write both minimal solid edges. For example, consider the differential biquiver $C^{e,b}(1,3)$. Then the differential of non-distinguished arrows is as follows:

$$\begin{aligned} \partial(e) = \partial(b) = \partial(v_a) &= \partial(v_f) = 0, \\ \partial(a) &= -b \cdot v_f \cdot e, \\ \partial(f) &= e \cdot v_a \cdot b, \\ \partial(v_b) &= -v_f \cdot e \cdot v_a, \\ \partial(v_e) &= v_a \cdot b \cdot v_f. \end{aligned}$$

Hence, there are 6 biquivers of type C and 12 differential biquivers.

- To define a differential biquiver on the diagram D it is sufficient to choose the *diagonal* of the configuration i.e. to choose the minimal and the maximal vertices. A differential biquiver of type D is denoted by $D(i,j)$, for $i,j \in I$. For example:

$$D(1,3) = \begin{tikzcd} {}^{2}\bullet \arrow[d, "a"'] & \bullet^{3} \arrow[l, "e"'] \arrow[dl, "d"] \arrow[d, "f"] \\ {}_{1}\bullet & \bullet_{4} \arrow[l, "b"] \end{tikzcd}$$

There is a unique minimal edge. Differentials of all other non-distinguished arrows are determined by the completeness rule. There are 12 differential biquivers of type D.

Altogether we get 56 differential biquivers. It turns out that the automaton of brick-reduction contains all of them as states.

Automaton

Let us present a brick-reduction automaton. The set of internal states splits

$$\Gamma := \Gamma_A \cup \Gamma_B \cup \Gamma_C \cup \Gamma_D$$

where Γ_T, is a set of differential biquivers on the diagram $T \in \{A, B, C, D\}$ and its dual. The transitions are given on examples:

- poset A

$$\begin{tikzcd}
& A^+(1) \arrow[loop left, "a^+"] \arrow[loop above, "b^+"] \arrow[loop right, "(d^+)"] \arrow[dl, "a^-"'] \arrow[d, "b^-"] \arrow[dr, "d^-"] & \\
B^+(1,2) & B^+(1,4) & B^+(1,3)
\end{tikzcd}$$

- poset B

$$\begin{tikzcd}
D^+(1,2) & & B^+(1,3) \arrow[ll, "b^-"'] \arrow[rr, "a^-"] \arrow[dl, "a^+"'] \arrow[dr, "b^+"] & & D^+(1,4) \\
& C^{e,b}(1,3) & & C^{a,f}(1,3) &
\end{tikzcd}$$

- poset C

$$\begin{tikzcd}
& B^-(4,2) & \\
B^+(3,1) & C^{e,b}(1,3) \arrow[l, "b^+"'] \arrow[r, "e^+"] \arrow[u, "e^-"'] \arrow[d, "b^-"] & B^+(1,3) \\
& B^-(2,4) &
\end{tikzcd}$$

- poset D

$$\begin{tikzcd}
A^+(1) & D^+(1,3) \arrow[l, "d^-"'] \arrow[r, "d^+"] & A^-(3)
\end{tikzcd}$$

We do not give calculations confirming the reduction steps listed above, since they are obtained by straightforward calculations in accordance with Remark 6.7.5 and Proposition 6.7.6. However, to stress that the differential biquiver $C^{b,e}(1,3)$ is nonlinear we consider the transition $a^+ : B^+(1,3) \to C^{b,e}(1,3)$.

Example 7.6.1.

$$B^+(1,3) = \begin{tikzcd} 2 \bullet \arrow[r, "e"] \arrow[d, "a"'] & \bullet 3 \arrow[dl, "d"] \\ 1 \bullet & \bullet 4 \arrow[l, "b"] \arrow[u, "f"'] \end{tikzcd} \quad \text{with} \quad \begin{gathered} \partial(a) = \partial(b) = \partial(v_d) = \partial(v_e) = \partial(v_f) = 0 \\ \partial(e) = -v_d a \quad \partial(f) = -v_d b \\ \partial(d) = b v_f + a v_e \\ \partial(v_a) = v_e v_d \\ \partial(v_b) = v_f v_d. \end{gathered}$$

Recall that a^+ encodes brick-reduction on the arrow a with $s_2 > s_1$. Apply minimal edge reduction on a^+ and all possible regularizations. Fix notations

$$1 \mapsto 1 \oplus 0, \quad 2 \mapsto 0 \oplus 2 \quad \text{and} \quad a \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right).$$

For simplicity we omit tildes in equations, and write equalities only for arrows attached to vertices $0, 2, 3, 4$.

- $\partial(b) = 0$ implies $\left(\begin{smallmatrix} b_1 \\ b_0 \end{smallmatrix}\right) w_3 = \left(\begin{smallmatrix} w_1 & 0 \\ \xi & w_0 \end{smallmatrix}\right)\left(\begin{smallmatrix} b_1 \\ b_0 \end{smallmatrix}\right)$, thus new $\tilde{b} := b_0$ and $\tilde{\partial}(\tilde{b}) = 0$;
- $\partial(e) = -v_d a$ implies $(e_0 \ e_2)\left(\begin{smallmatrix} w_0 & 0 \\ \eta & w_2 \end{smallmatrix}\right) = w_3(e_0 \ e_2) - (v_{d1} \ v_{d0})\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ hence, $\tilde{e} := e_2$ and $\tilde{\partial}(\tilde{e}) = 0$, e_{30} is superfluous and $v_{d_0} = -e_2\eta = -\tilde{e} \cdot \tilde{v}_{\tilde{a}}$, where $\tilde{v}_{\tilde{a}} := \eta$;
- $\partial(f) = -v_d b$ implies $f w_4 = w_3 f - (v_{d1} \ v_{d0})\left(\begin{smallmatrix} b_1 \\ b_0 \end{smallmatrix}\right)$, cutting out terms attached to the vertex 1 for $\tilde{f} := f$ obtain $\tilde{\partial}(\tilde{f}) = v_{d_0} \cdot b_{04} = \tilde{e} \cdot \tilde{v}_{v\tilde{a}} \cdot \tilde{b}$;
- $\partial(d) = b v_f + a v_e$ implies $\left(\begin{smallmatrix} d_1 \\ d_0 \end{smallmatrix}\right) w_3 = \left(\begin{smallmatrix} w_1 & 0 \\ \xi & w_0 \end{smallmatrix}\right)\left(\begin{smallmatrix} d_1 \\ d_0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} b_1 \\ b_0 \end{smallmatrix}\right) v_f + \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} v_{e0} \\ v_{e2} \end{smallmatrix}\right)$ arrow $\tilde{d}$ is superfluous and $v_{e_0} = -\tilde{b} \cdot \tilde{v}_{\tilde{f}}$;
- for the distinguished loop of the vertex 2 we have $\partial(a_2) = -v_a a - v_e e$ hence $\left(\begin{smallmatrix} a_0 & a_{02} \\ a_{20} & a_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} w_0 & 0 \\ \eta & w_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} w_0 & 0 \\ \eta & w_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_0 & a_{02} \\ a_{20} & a_2 \end{smallmatrix}\right) - \left(\begin{smallmatrix} u_{01} & u_0 \\ u_{21} & u_{20} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} v_{e_0} \\ v_{e_2} \end{smallmatrix}\right)(e_0 \ e_2)$ thus $\tilde{a} := a_{02}$ obtain $\tilde{\partial}(\tilde{a}) = v_{e_0} e_2 = -\tilde{b} \cdot \tilde{v}_{\tilde{f}} \cdot \tilde{e}$ and $\tilde{a}_2$ is the new distinguished loop.

Analogously we get differentials for dotted arrows. Omitting tildes, we obtain a BT-differential biquiver

$$C^{b,e}(1,3) = \begin{tikzcd} 2 \bullet \arrow[r, "e"] \arrow[d, "a"'] & \bullet 3 \\ 1 \bullet & \bullet 4 \arrow[l, "b"] \arrow[u, "f"'] \end{tikzcd} \quad \text{with} \quad \begin{gathered} \partial(b) = \partial(e) = \partial(v_a) = \partial(v_f) = 0, \\ \partial(a) = -b v_f e \\ \partial(f) = e v_a b \\ \partial(v_b) = -v_f e v_a \\ \partial(v_e) = v_a b v_f. \end{gathered}$$

From now our goal is to present the automaton and describe equivalent paths on it. We start with the state $A^+(1)$. From the calculations above we obtain the following fragments of the automaton:

$$
\begin{tikzcd}
 & & B^+(1,2) \arrow[dll, "d^-"'] \arrow[drr, "b^-"] & & \\
D(1,4) \arrow[drr, "b^-"] & & & & D(1,3) \arrow[dll, "d^-"'] \\
 & & A^+(1) \arrow[uu, "a^-"] \arrow[dll, "d^-"'] \arrow[drr, "b^-"] & & \\
B^+(1,3) \arrow[uu, "a^-"] \arrow[drr, "b^-"'] & & & & B^+(1,4) \arrow[uu, "a^-"'] \arrow[dll, "d^-"] \\
 & & D(1,2) \arrow[uu, "a^-"] & &
\end{tikzcd}
$$

Figure 7.6.1

The following lemma is obvious.

Lemma 7.6.2.
- *Arbitrary arrows* $m^\sigma, n^\sigma : A^\sigma(i) \to A^\sigma(i)$ *commutes*
$$m^\sigma n^\sigma \sim n^\sigma m^\sigma;$$
- *any two paths* p_1 *and* $p_2 : A^\sigma(i) \to A^\sigma(i)$ *of length three are equivalent. (Commutation of length three)*

Analogously, consider the the partial automaton around the differential biquiver of type C.

$$
\begin{tikzcd}
D(3,4) & B^-(4,2) \arrow[l, "b^+"'] \arrow[r, "f^+"] \arrow[d, shift left, "f^-"] & D(1,4) \\
B^+(3,1) \arrow[u, "e^-"] \arrow[d, "f^-"'] \arrow[r, shift left, "f^+"] & C^{e,b}(1,3) \arrow[l, shift left, "b^+"] \arrow[u, shift left, "e^-"] \arrow[r, shift left, "e^+"] \arrow[d, shift left, "b^-"] & B^+(1,3) \arrow[l, shift left, "a^+"] \arrow[u, "a^-"'] \arrow[d, "b^-"] \\
D(3,2) & B^-(2,4) \arrow[l, "a^+"'] \arrow[r, "e^+"] \arrow[u, shift left, "a^-"] & D(1,2)
\end{tikzcd}
$$

Figure 7.6.2

For better visualization we glue state $C^{e,b}(1,3)$ together with $C^{a,f}(1,3)$. However, by this identification, we should be careful and remember that there are no paths $m^\sigma n^\delta$ for $m = n$ and for (m, n) being a pair of on the opposite edges. Such pairs are: a and f; b and e; and c and d.

We have

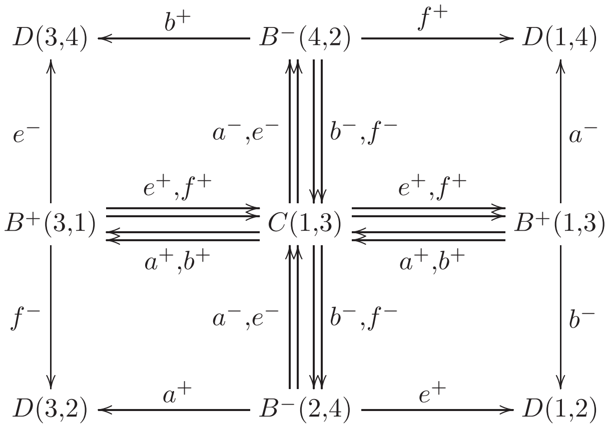


Figure 7.6.3

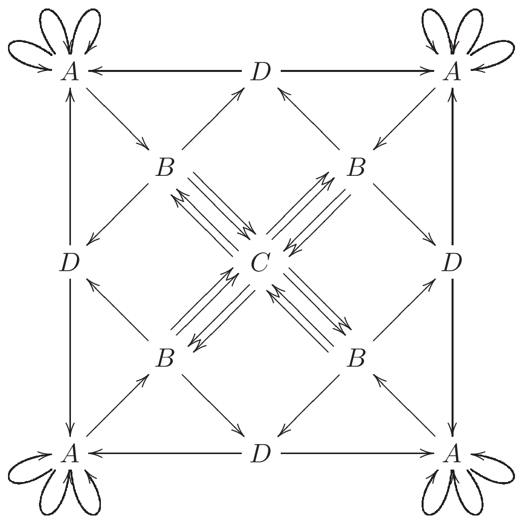
Lemma 7.6.3. *Let $m^\sigma n^\delta$ and $n^\delta\ m^\sigma$ be two paths with common source and target then*

$$m^\sigma n^\delta \sim n^\delta m^\sigma.$$

Proof. For paths touching a configuration A the statement follows from Lemma 7.6.2. The other such paths are $m^\sigma n^\sigma$ from $B^\sigma(i,j)$ to $B^\sigma(j,i)$; and $m^\sigma n^{-\sigma}$ connecting configuration C and D (see figure 7.6.2). For both cases the statement is trivial. $\square$

Brick-reduction automaton

The automaton of reduction can be realized on a cube: a configuration A on a vertex, a configurations D on an edge. Then on each facet, we get:



Let us give the whole automaton of the brick-reduction: Note that each state A has three loops and there are no paths xx and xy through $C(i,j)$, where x and y are lying on opposite edges.

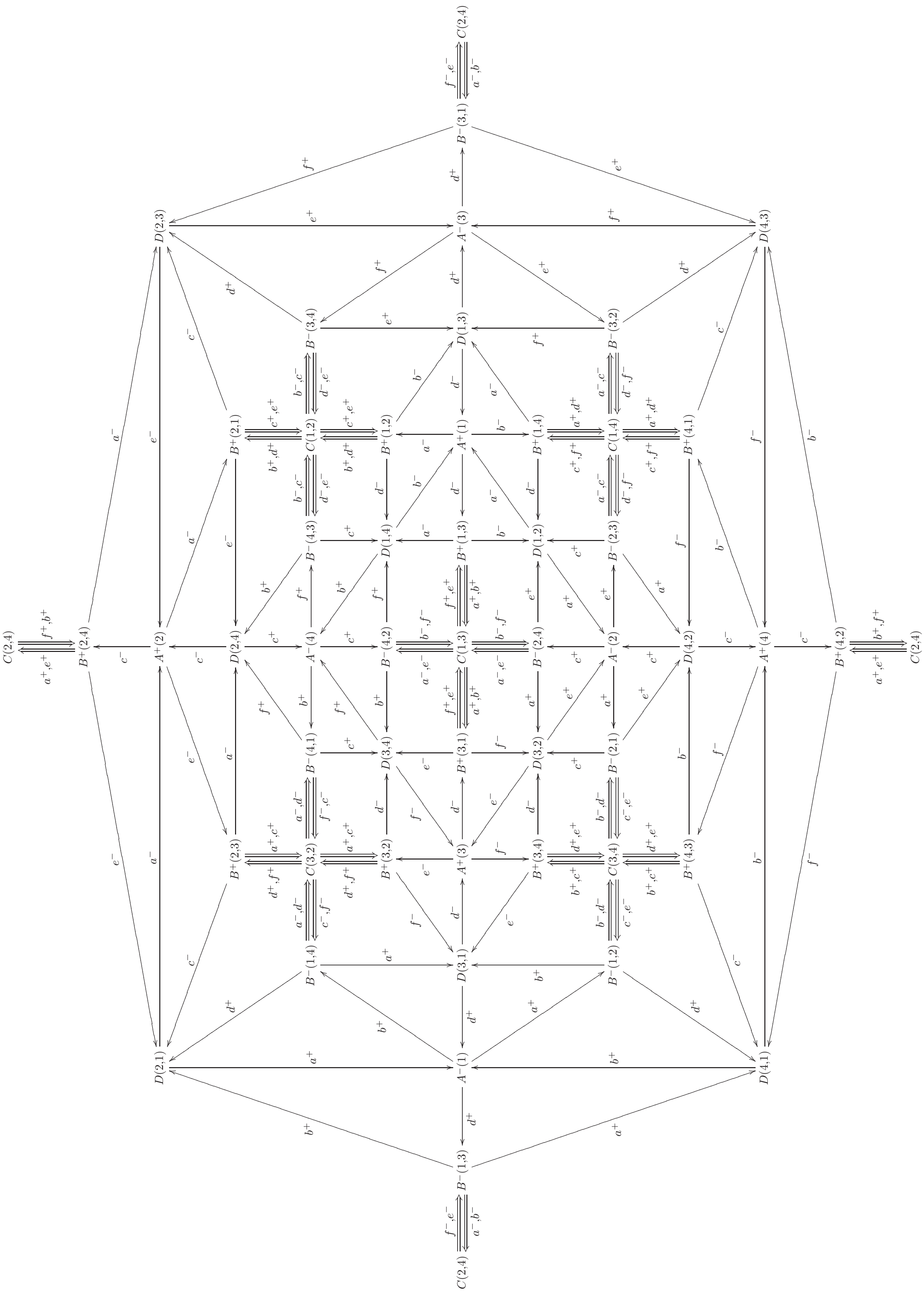


Figure 7.6.4

Rank and degree

For configurations of types A and B and a vector dimension $\mathbb{s} \in \mathbb{N}^4$ let us introduce *rank* ρ and *degree* δ of $\mathbb{s}$:

- for a configuration $A^\sigma(i)$ define $\rho := s_1 + s_2 + s_3 + s_4$ and $\delta := s_i$;
- for a configuration $B^\sigma(i,j)$ take $\rho := s_i + 2s_j + s_l + s_k$ and $\delta := s_i + s_j$, where $\{l,k,i,j\} = I$.

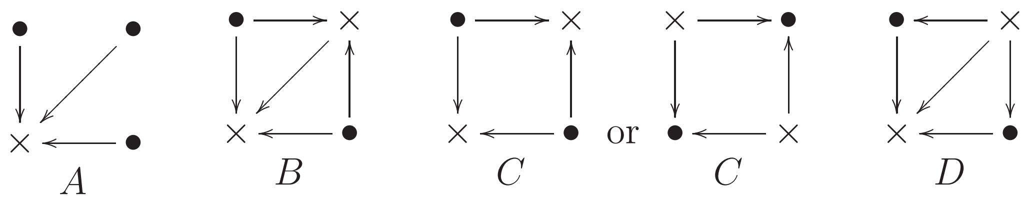
It is useful to introduce an subsidiary parameter $\alpha := \rho - \delta$.

Configurations of types A and B are *principal* in sense that they encode the problem of classification of vector bundles. Note that if $A^\sigma(i)$ or $B^\sigma(i,j)$ with a vector dimension $\mathbb{s}$ encode a matrix problem for $\mathsf{VB}^s_E(r, \mathbb{d})$ then $(r,d) = (\rho, \delta)$.

Splitting configurations

In this subsection we analyze what happens at the final steps of the brick-reduction i.e. if some sizes are zero.

A configuration is called *splitting* if it fulfill the property: if some sizes s_i are zero then it becomes disconnected. The list of the splitting configurations is as follows:



where "$\times$" denotes a vertex i with $s_i = 0$. Let us analyze how such configurations can appear.

A: A splitting configuration of type A appears from the configuration B along a path of length 2. For example: the configuration $B^+(1,3)$ with a tuple of sizes $(s_1,s_2,s_3,s_4) = (x+z, x, y, z)$ can be reduced along the path $B^+(1,3) \xrightarrow{a^-b^-} A^+(1)$ then the new vector dimension is $\mathbb{s}' = (0,x,y,z)$.

B: A splitting configuration of type B can not appear in the course of brick-reduction. For example: the splitting configuration $B^+(1,3)$ appears from $A^+(1)$ along the arrow d^- or from $C(1,3)$ along f^+ or e^+; but neither these arrows change sizes s_1, s_3.

C,D: Splitting configurations of types C and D can appear from a configuration of type B. For example: the configuration $B^+(1,3)$ with a tuple of sizes $(s_1,s_2,s_3,s_4) = (x,0,y,x)$ can be reduced along both arrows b^+ and b^- :

$$C^{a,f}(1,3) \xleftarrow{b^-} B^+(1,3) \xrightarrow{b^+} D(1,2).$$

The vector dimension for $C^{a,f}(1,3)$ is $\mathbb{s}' = (0,0,y,x)$ and for $D(1,2)$ is $\mathbb{s}' = (x,0,y,0)$.

Let γ be a fixed configuration of type A or B and $\mathbb{s}$ be a fixed vector dimension. Note that this set up leads to a splitting configuration if and only if its rank ρ and degree δ have a common divisor. This claim follows from the following lemma.

Lemma 7.6.4. *Let γ and $\gamma' \in \Gamma_A \cup \Gamma_B$, $p : \gamma \longrightarrow \gamma'$ be a fixed path and $\mathbb{s} \in \mathbb{N}^4$ be a fixed tuple of sizes. Then for $\mathbb{s}' = p(\mathbb{s})$ it holds*

$$g.c.d.(\rho, \delta) = g.c.d.(\rho', \delta'),$$

where ρ and δ are rank and degree of $\mathbb{s}$ in γ and ρ' and δ' are rank and degree of $\mathbb{s}'$ in γ'.

Proof. We check the property for the shortest paths, since all other paths can be obtained by concatenation of the shortest ones.

1. $A^\sigma(i) \xrightarrow{p} A^\sigma(i)$ a loop of length one: $(\rho, \delta) \mapsto (\rho - \delta, \delta)$ and $(\alpha, \delta) \mapsto (\alpha - \delta, \delta)$,
   $A^\sigma(i) \xrightarrow{p} A^\sigma(i)$ the loop of length three: $(\rho, \delta) \mapsto (\rho - 3\delta, \delta)$ and $(\alpha, \delta) \mapsto (\alpha, \delta - \alpha)$;

2. $A^\sigma(i) \xrightarrow{p} B^\sigma(i,j)$ a path of length one: $(\rho, \delta) \mapsto (\rho, \delta)$ and $(\alpha, \delta) \mapsto (\alpha, \delta)$;

3. paths of length two: $B^\sigma(i,j) \xrightarrow{p} A^\sigma(i)$ $(\rho, \delta) \mapsto (\delta, 2\delta - \rho)$ and $(\alpha, \delta) \mapsto (\alpha, \delta - \alpha)$;
   $B^\sigma(i,j) \xrightarrow{p} A^{-\sigma}(k)$ $(k \neq i,j)$: $(\rho, \delta) \mapsto (\rho - \delta, \rho - 2\delta)$ and $(\alpha, \delta) \mapsto (\alpha, \alpha - \delta)$;

4. paths of length two: $B^\sigma(i,j) \xrightarrow{p} B^\sigma(i,j)$: $(\rho, \delta) \mapsto (\rho - \delta, \delta)$ and $(\alpha, \delta) \mapsto (\alpha - \delta, \delta)$;
   $B^\sigma(i,j) \xrightarrow{p} B^\sigma(j,i)$: $(\rho, \delta) \mapsto (\rho - \delta, \delta)$ and $(\alpha, \delta) \mapsto (\alpha - \delta, \delta)$;
   $B^\sigma(i,j) \xrightarrow{p} B^{-\sigma}(l,k)$ $(l,k \neq i,j)$: $(\rho, \delta) \mapsto (\rho - \delta, \rho - 2\delta)$ and $(\alpha, \delta) \mapsto (\delta, \alpha - \delta)$.

$\square$

The lemma implies an obvious corollary:

Lemma 7.6.5. *Following notations in Lemma 7.6.4 if there exists a path p taking a set-up $\gamma \in \Gamma_A \cup \Gamma_B$ and $\mathbb{s} \in \mathbb{N}^4$ into a splitting configuration then its rank ρ and degree δ have a common divisor.*

To make the statement rigorous we formulate it as a theorem:

Theorem 7.6.6. *Let (Q, ∂) be a differential biquiver corresponding to the configuration of type A or B. Then $\mathsf{Br}(Q, \partial)(\mathbb{s})$ is not empty if and only if*

$$g.c.d.(\rho, \delta) = 1.$$

Proof. Let γ be a fixed configuration of type A or B and $\mathbb{s} \in \mathbb{N}^4$ be a tuple such that $g.c.d.(\rho, \delta) = 1$. Lemma 7.6.5 asserts that on each step the new set up $(\gamma', \mathbb{s}')$ is non-splitting. We reduce $\mathbb{s}$ along a path on the automaton until $(\rho, \delta) = (1, 0)$. Hence, in course of brick-reduction we obtain a problem of dimension one, which is non-empty. Moreover, since (Q, ∂) is a full BT-differential biquiver, it contains one one-parameter family of bricks. □

Reformulating this theorem in terms of vector bundles we obtain the claim of Theorem 5.0.1 for the Kodaira fiber IV.

Principal Automaton

The principal automaton on states of type A and B similarly to the brick-reduction automaton should be regarded on a cube. Configurations A are placed on its vertices and each facet contains four configurations of type B. We present a diagram of one facet:

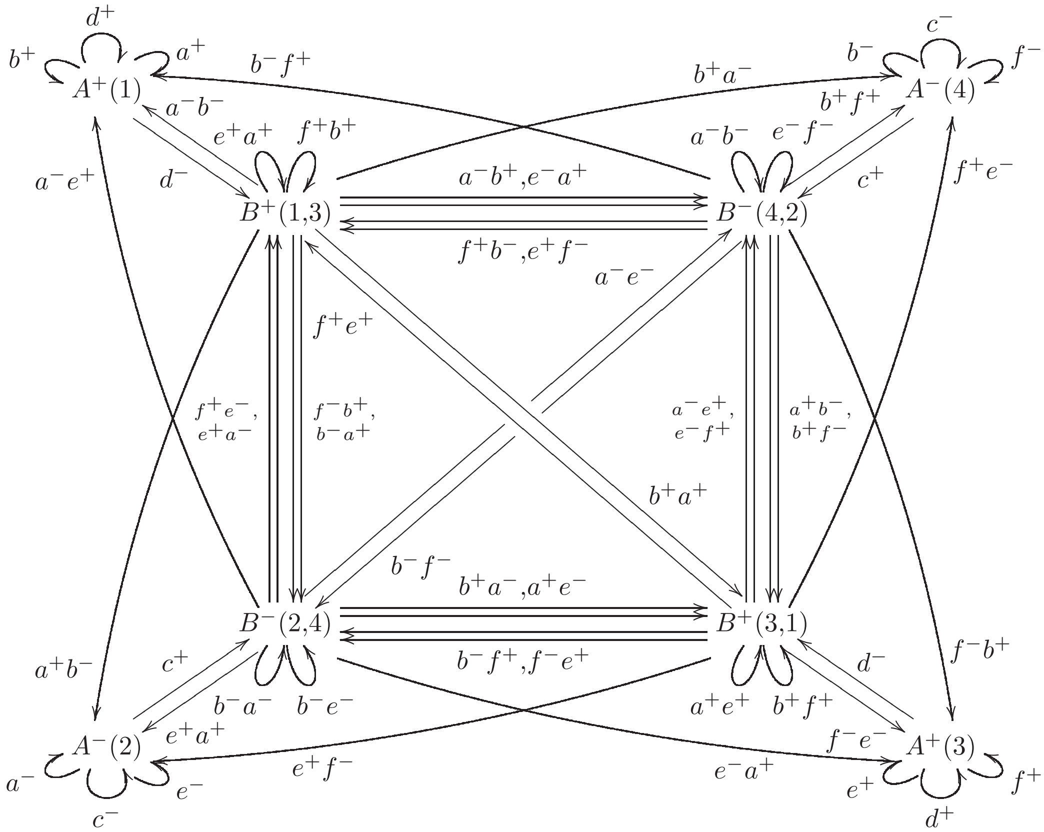


Figure 7.6.5 A facet of the principal automaton.

Algorithm

Assume we have a state γ of type A or B and a tuple of sizes $\mathbb{s} \in \mathbb{N}^4$ such that $g.c.d.(\rho, \delta) = 1$. In order to find a canonical form we proceed as follows: we take path $p : \gamma \to \gamma'$ on the principal automaton reducing a tuple $\mathbb{s}$ to $\mathbb{s}'$ such that $(\rho', \delta') = (1, 0)$. Input a canonical form of size one (usually $\lambda \in \mathbb{k}$) on the state γ' and construct the canonical form inductively along the reversed path p^*.

Note that the canonical form K of a brick obtained by matrix reduction along a path p depends on p. However, canonical forms of a vector dimension $\mathbb{s}$ and a fixed differential biquiver γ constructed along arbitrary paths p are equivalent.

Equivalent paths

Let us describe equivalent paths. We should mention that this time besides the commutation relations listed in Lemma 7.6.3 there exists some nontrivial equivalences. Let us illustrate this phenomena with some examples.

Example 7.6.7. Note that starting from the state $A^+(1)$ we get:

$$(e^+a^+)d^- \sim d^-a^+ \text{ and } (f^+b^+)d^- \sim d^-b^+.$$

Starting from $B^+(1,3)$ we get

$$(b^+a^-)(f^+b^+) \sim (b^+f^+c^+)(b^+a^+) \text{ and } (a^+b^-)(e^+a^+) \sim (a^+c^+e^+)(a^+b^-).$$

Analogous relations hold for all states of types A and B.

Example 7.6.8. Let us describe paths $A^+(1) \xrightarrow{p} A^-(4)$ of length three and five. All of them can be seen on the following partial automaton:

$$\begin{tikzcd}
 & & C(1,2) \arrow[r, shift left, "d^+"] \arrow[r, shift right, "b^+"'] \arrow[d, shift right, "d^-"'] \arrow[d, shift left, "e^-"] & B^+(2,1) \arrow[d, "e^-"] & \\
 & B^+(1,2) \arrow[ur, shift left, "b^+"] \arrow[ur, shift right, "d^+"'] \arrow[dr, "d^-"] & B^-(4,3) \arrow[r, "b^+"] \arrow[d, "c^+"] & D(2,4) \arrow[dr, "c^+"] & \\
A^+(1) \arrow[ur, "a^-"] \arrow[dr, "d^-"'] & & D(1,4) \arrow[rr, "b^+"] & & A^-(4) \\
 & B^+(1,3) \arrow[ur, "a^-"] \arrow[dr, shift left, "b^+"] \arrow[dr, shift right, "a^+"'] & B^-(4,2) \arrow[u, "f^+"] \arrow[r, "b^+"] & D(3,4) \arrow[ur, "f^+"'] & \\
 & & C(1,3) \arrow[u, shift left, "e^-"] \arrow[u, shift right, "a^-"'] \arrow[r, shift left, "a^+"] \arrow[r, shift right, "b^+"'] & B^+(2,1) \arrow[u, "e^-"] &
\end{tikzcd}$$

There exists a unique path of length three

$$p_0 := b^+a^-d^- \sim b^+d^-a^-$$

and three nonequivalent paths of length five:

$$\begin{aligned}
p_1 &:= c^+e^-d^+b^+a^- \sim c^+e^-b^+d^+a^- \sim c^+b^+e^-d^+a^- \sim b^+c^+e^-d^+a^-;\\
p_2 &:= f^+e^-a^+b^+d^- \sim f^+e^-b^+a^+d^- \sim f^+b^+e^-a^+d^- \sim b^+f^+e^-a^+d^-;\\
p_3 &:= c^+b^+d^-b^+a^- \sim b^+c^+d^-b^+a^- \sim f^+b^+a^-b^+d^- \sim b^+f^+a^-b^+d^-.
\end{aligned}$$

Note that the circled equivalence relation can no be obtained by commutations, since the left and the right parts contain different arrows.

To have a general picture for paths of length five from connecting configurations of type A let us consider another example. However, note that here all equivalences are just consequences of commutation relations.

Example 7.6.9. Let us describe paths $A^+(1) \xrightarrow{p} A^+(3)$ of length five. All of them can be seen on the following partial automaton:

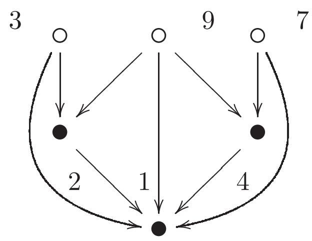
$$\begin{tikzcd}
 & & & B^-(4,2) \arrow[dr, "b^+"] & & \\
 & & C(1,3)^{e,b} \arrow[ur, "e^-"] \arrow[dr, "b^+"] \arrow[dddr, "b^-"] & & D(1,3)^e, b \arrow[dr, "f^-"] & \\
A^+(1) \arrow[r, "d^-"] & B^+(1,3) \arrow[ur, "a^+"] \arrow[dr, "b^+"'] & & B^+(3,1) \arrow[ur, "e^-"] \arrow[dr, "f^-"] & & A^+(3) \\
 & & C(1,3)^{a,f} \arrow[uuur, "a^-"] \arrow[ur, "a^+"] \arrow[dr, "f^-"'] & & D(1,3)^e, b \arrow[ur, "e^-"'] & \\
 & & & B^-(2,4) \arrow[ur, "a^+"'] & &
\end{tikzcd}$$

Thus we have three nonequivalent paths of length five

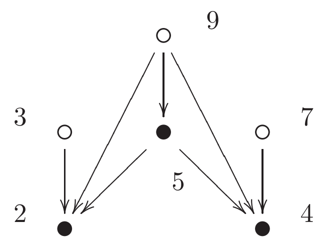
$$\begin{aligned}
w_1 &:= f^-b^+a^-b^+d^-;\\
w_2 &:= e^-a^+b^-a^+d^-;\\
w_3 &:= f^-b^+e^-a^+d^- \sim f^-e^-b^+a^+d^- \sim f^-e^-a^+b^+d^- \sim\\
&\qquad e^-f^-a^+b^+d^- \sim e^-a^+f^-b^+d^- \sim e^-f^-b^+a^+d^-.
\end{aligned}$$

7.7 Matrix problem for torsion free sheaves on a tacnode curve

In this section we consider matrix problems BM_P formulated in Section 4.5 for torsion free sheaves on a tacnode curve. We use the system of short notations introduced in Section 7.2. Note that $\mathsf{BM}_P \cong \mathsf{Rep}(Q, \partial)$, where the differential biquiver (Q, ∂) is encoded by the poset P. Recall that the poset P has form either



or



Both biquivers looks quite complicated, however there is one simplifying consideration. Assume M is a brick sincere at at least two non-distinguished vertices i and j. Since in course of the reduction the dimensions of M_i and M_j remain the same thus on some step we obtain a fragment $\circ \xrightarrow{b} \circ$ with the minimal edge b. Then by Lemma 6.7.3 the dotted arrow v_b is not involved in any differential, hence we get a contradiction and thus

$$\mathsf{Br}(Q, \partial) = \mathsf{Br}(Q^9, \partial) \cup \mathsf{Br}(Q^7, \partial) \cup \mathsf{Br}(Q^3, \partial),$$

where $Q^3 = Q|_{I\setminus\{7,9\}}$, $Q^7 = Q|_{I\setminus\{3,9\}}$ and $Q^9 = Q|_{I\setminus\{3,7\}}$. Moreover, since each size s_3, s_7 or s_9 is smaller than any other, thus each of them is not grater than 1.

Here we consider the case $Q = Q_9$, since only this one correspond to a torsion free sheaves with the same rank on each component. (The problem for sheaves with different ranks will be considered elsewhere).

Rank and degree

Note that a configuration $D(1,9)$ has a unique minimal edge $1 \to 9$ with $s_9 \leq s_1$. Thus there exists a unique step of reduction $D(1,9) \longrightarrow A^+(1)$. Note that the rank and degree (ρ, δ) defined for these configurations show the best correlation with usual rank and degree.

- For $\mathsf{TF}^s_E(r, d_1, d_2)$ with $r > d$ (recall that in this situation $d = d_1 + d_2 + 1$) we get the problem $D(1,9)$ with sizes $\mathbb{s} = (s_2, 1, s_4, s_5)$, where $d_1 = s_4$, $d_2 = s_2$, and $r = s_1 + s_2 + s_4$. Since $D(1,9) \longrightarrow A^+(1)$ with $\mathbb{s} \mapsto \mathbb{s}' = (s_1 - 1, s_2, 1, s_4)$, we have

$$(\rho, \delta) = (s'_1 + s_2 + s_4 + 1, s'_1) = (s_1 + s_2 + s_4, s_1 - 1) = (r, r - d).$$

- For $\mathsf{TF}^s_E(r, d_1, d_2)$ with $d > r$ we get the problem $B^-(9,5)$ on the set of vertices $\{2, 4, 5, 9\}$ with sizes $\mathbb{s} = (s_5, s_2, 1, s_4)$, where $d_1 = s_4 + s_5$, $d_2 = s_2 + s_5$, and $r = s_2 + s_4 + s_5$. Thus

$$(\rho, \delta) = (s_2 + s_4 + 1 + s_5, s_5 + s_9) = (d, d - r).$$

Changing notations of vertices "9" by "3" and "5" by "1" if needed, we get configurations:

$$\begin{tikzcd} {}^{2}\bullet \arrow[d] & \circ^{3} \arrow[l] \arrow[d] \arrow[dl] \\ {}_{1}\bullet & \bullet_{4} \arrow[l] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {}^{2}\bullet & \circ^{3} \arrow[l] \arrow[d] \arrow[dl] \\ {}_{1}\bullet \arrow[u] \arrow[r] & \bullet_{4} \end{tikzcd} \tag{7.25}$$

$$D(1,3) \qquad\qquad B^-(3,1).$$

The automaton of brick-reduction is a sub-automaton on 7.6.4. From the very beginning we assume $s_3 = 1 \leq s_1, s_2, s_4$. Hence, some arrows on 7.6.4 do not exist. Analogously, to the case of torsion free sheaves on a cuspidal cubic curve, we take to consideration not all possible reductions to avoid configurations of types A^- and B^+, which can not be interpreted in terms of torsion free sheaves.

Splitting configurations

Assume we start with a configuration γ of type A or B. Since for a torsion free sheaf we assume $s_3 = 1$ thus the splitting configurations are $A^\sigma(i)$ and $D(i,j) = C(i,j)$ for $i, j \in \{1, 2, 4\}$. As was shown in the previous section they appear only if ρ and δ of γ have a common divisor, Theorem 7.6.6 implies that there exists a brick only if $g.c.d.(\rho, \delta) = 1$. Since in this case the differential biquiver is of BT-type but not full, thus for a given vector dimension the brick (if it exists) is unique. Reformulating this in terms of rank and degree we obtain the statement of Theorem 4.0.1.

Principal states

The configurations of 7.6.4 which can be interpreted as torsion free sheaves on a tacnode curve are $D(i, 3)$ and $B^-(3, i)$ for $i \in \{1, 2, 4\}$. Note that a configuration $D(i, 3)$ has a unique minimal edge $i \to 3$ with $s_3 \leq s_i$. Thus there exists a unique step of reduction

$$D(i, 3) \longrightarrow A^+(i).$$

Principal automaton

The course of reduction $\mathsf{TF}^s_E(r, \mathbb{d}) \to \mathsf{TF}^s_E(r', \mathbb{d}')$ with $r' < r$ can be encoded as a path on the following principal automaton:
The equivalence relations are the same as for the automaton 7.5.2. Thus we get the uniform description for vector bundles and torsion free sheaves on a tacnode curve.

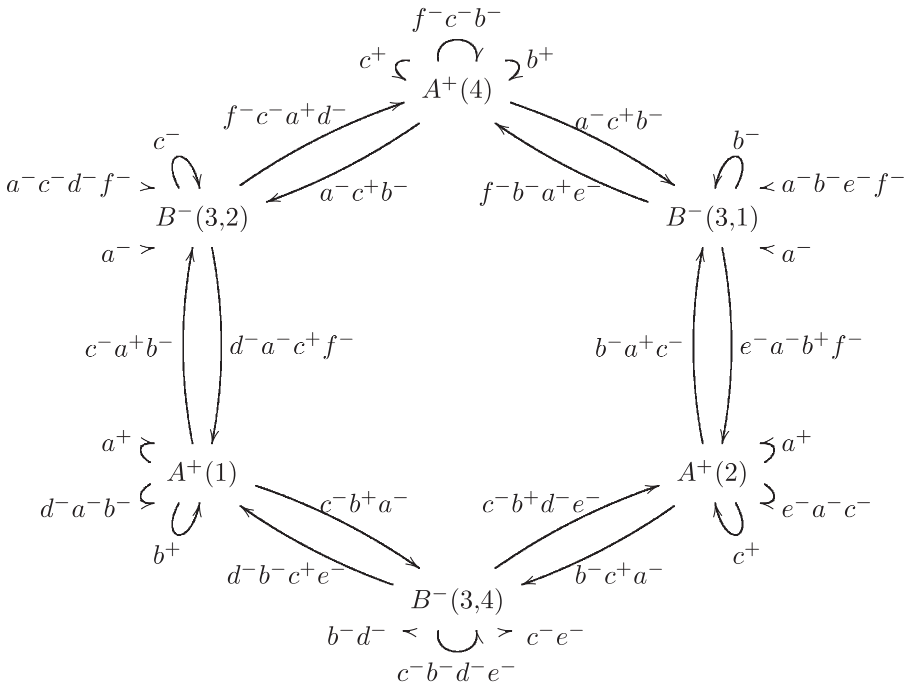


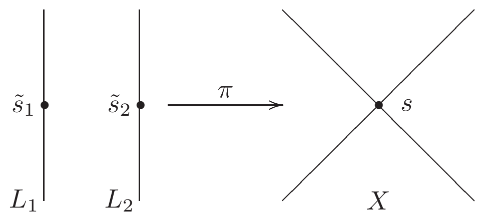
Figure 7.7.1 Principal reduction automaton for torsion free sheaves on a tacnode curve.

Appendix A

Vector bundles on curves of arithmetic genus zero

Transversal intersection of two projective lines

Let us illustrate the method of triples and matrix problems introduced in Section 2.1 on projective curves of arithmetic genus zero. The simplest example one can imagine is a transversal intersection of two projective lines locally given by the equation $xy = 0$:



Let us describe vector bundles on X.

Choose coordinates $(z_0 : z_1)$ on each component of the normalization $\widetilde{\mathrm{X}} \cong L_1 \sqcup L_2 \xrightarrow{\pi} X$ such that the preimages $\tilde{s}_1$ and $\tilde{s}_2$ of the singular point s of X have coordinate $0 := (0 : 1)$ on each component. Let $U_k := \{(z_0 : z_1) | z_1 \neq 0\}$, $k = 1, 2$, be affine neighborhoods of $(0 : 1)$ on each component, U a disjoint union of U_1 and U_2. Introduce local coordinates $x := z_0/z_1$ and $y := z_0/z_1$ on L_1 and L_2 respectively. The normalization sheaf splits $\widetilde{\mathcal{O}} = \mathcal{O}_{L_1} \oplus \mathcal{O}_{L_2}$ and $\widetilde{\mathcal{O}}(U) = \Bbbk[x] \oplus \Bbbk[y]$. Locally at s it holds $\mathcal{J}_s = \langle x, y\rangle$ and hence, $\mathcal{O}_{\widetilde{S}} = \Bbbk(\tilde{s}_1) \oplus \Bbbk(\tilde{s}_1)$, $\mathcal{O}_S = \Bbbk(s)$. For a triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ we fix:

- a decomposition $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}|_{L_1} \oplus \widetilde{\mathcal{F}}|_{L_2} \cong \Big(\bigoplus\limits_{n\in\mathbb{Z}} \mathcal{O}_{L_1}(n)^{r(n,1)}\Big) \oplus \Big(\bigoplus\limits_{n'\in\mathbb{Z}} \mathcal{O}_{L_2}(n')^{r(n',2)}\Big)$, where $\sum\limits_{n\in\mathbb{Z}} r(n,1) = \sum\limits_{n'\in\mathbb{Z}} r(n',2) = r$;

- an isomorphism $\mathcal{M} \cong \big(\Bbbk(s)\big)^r$;

- note, that the choice of coordinates on on each component L_k, $k = 1, 2$, fixes two canonical sections z_0 and z_1 of $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ and we fix the following trivializations

$$\begin{aligned} \mathcal{O}_{L_k}(n) \otimes \Bbbk(\tilde{s}_k) &\xrightarrow{\sim} \Bbbk(\tilde{s}_k) \\ \zeta \otimes 1 &\longmapsto \tfrac{\zeta}{z_1^n}(0:1), \end{aligned}$$

chosen isomorphisms only depends on the choice of coordinates of L_k.

In such a way we supply $\tilde{\imath}^*\widetilde{\mathcal{F}}$ with a basis and get an isomorphism

$$\tilde{\imath}^*\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}|_{L_1}(\tilde{s}_1) \oplus \widetilde{\mathcal{F}}|_{L_2}(\tilde{s}_2) \xrightarrow{\cong} \Bbbk^r(\tilde{s}_1) \oplus \Bbbk^r(\tilde{s}_2),$$

and hence, maps $\tilde{\mu}$, $\tilde{\imath}^*F$ and $\tilde{\pi}^*f$ can be written as matrices.

- The map $\tilde{\mu} : \Bbbk^r(s) \to (\Bbbk^r(\tilde{s}_1), \Bbbk^r(\tilde{s}_2))$, is given by two matrices

$$\tilde{\mu} = (\mu_1(0), \mu_2(0)) \in \mathrm{GL}(\Bbbk, r) \times \mathrm{GL}(\Bbbk, r);$$

- For each component L_k, if we have a morphism $\mathcal{O}_{L_k}(n) \to \mathcal{O}_{L_k}(m)$ given by a homogeneous form $Q(z_0, z_1)$ of degree $m-n$, then the induced map $\mathcal{O}_{L_k}(n) \otimes \mathcal{O}_{L_k}/\widetilde{\mathcal{J}} \longrightarrow \mathcal{O}_{L_k}(m) \otimes \mathcal{O}_{L_k}/\widetilde{\mathcal{J}}$ is given by the map $pr(Q(z_0, z_1)/z_1^{m-n}) = Q(0:1)$. Hence, for any endomorphism (F, f) of the triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\mu})$ the induced map $\tilde{\imath}^*F = (\tilde{\imath}^*F_1, \tilde{\imath}^*F_2) : \tilde{\imath}^*\widetilde{\mathcal{F}} \longrightarrow \tilde{\imath}^*\widetilde{\mathcal{F}}$ has the form

$$\tilde{\imath}^*F = (F_1(0), F_2(0)).$$

  Here we write $(F_1(0), F_2(0))$ instead of $(F_1(0:1), F_2(0:1))$ in order to simplify the notations. Each of the matrices $F_1(0)$ and $F_2(0)$ is a lower-block-triangular matrix consisting of blocks $F_{mn} \in \mathrm{Mat}_{\Bbbk}(r_m \times r_n)$, for $m \geq n$. The morphism F is invertible, if and only if diagonal blocks F_{nn} of both matrices belong to $\mathrm{GL}(\Bbbk, r_n)$.

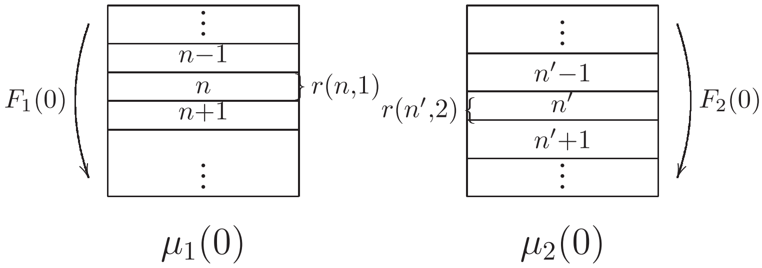
- Induced map $\tilde{\pi}^*f = (f, f)$ belongs to the diagonal of $\mathrm{Mat}_{\Bbbk}(r \times r) \times \mathrm{Mat}_{\Bbbk}(r \times r)$ and to the diagonal of $\mathrm{GL}(\Bbbk, r) \times \mathrm{GL}(\Bbbk, r)$ if it is invertible.

We obtain the following matrix transformations:

$$(\mu_1(0), \mu_2(0)) \mapsto (F_1(0)\mu_1(0)f^{-1}, F_2(0)\mu_2(0)f^{-1}). \tag{A.1}$$

Matrix problem for transversal intersection of two lines

We have two invertible matrices $\mu_1(0)$ and $\mu_2(0)$ of the same size. Each of them is independently divided into horizontal blocks. We label horizontal blocks of $\mu_1(0)$ and horizontal blocks of $\mu_2(0)$ by integers, such labels of blocks are called *weights*. Thus we obtain a matrix problem of the following form



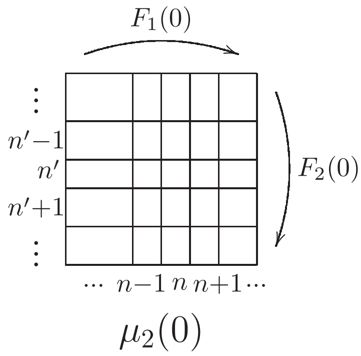
We are allowed to perform only the following transformations:

1. An arbitrary invertible elementary transformation of columns simultaneous for $\mu_1(0)$ and $\mu_2(0)$. This is the transformation f as in (A.1).

2. In any of the matrices $\mu_1(0)$ or $\mu_2(0)$ we can independently perform any invertible elementary transformation of rows inside of a block n. Such transformations correspond to diagonal blocks (F_{nn}) of the matrices $F_1(0)$ or $F_2(0)$.

3. In any of the matrices $\mu_1(0)$ or $\mu_2(0)$ we can add independently a scalar multiple of any row with a lower weight to any row with a higher weight. Such transformations correspond to blocks (F_{mn}) with $m > n$ of the matrices $F_1(0)$ or $F_2(0)$

If one of the matrices $\mu_1(0)$ or $\mu_2(0)$, say $\mu_1(0)$, is reduced to the identity form, then for the other one the matrix transformations left are row transformations $F_2(0)$ and column transformations $f = F_1(0)$. Thus, we obtain new partition of the matrix $\mu_2(0)$ into vertical blocks and the following matrix problem:



1. We can do any invertible elementary transformations of columns inside of any column block n. Such transformations correspond to diagonal blocks (F_{nn}) of F_1.

2. We can add a scalar multiple of any column of weight n to any column of a bigger weight m. Such transformations correspond to blocks (F_{mn}) with $m > n$ of the matrix $F_1(0)$.

3. We can do any invertible elementary transformations of rows inside of any block n' Such transformations correspond to diagonal blocks $(F_{n'n'})$ of $F_2(0)$.

4. We can add a scalar multiple of any row of weight n' to any row of higher weight m'. Such transformations correspond to blocks $(F_{m'n'})$ with $m' > n'$ of the matrix $F_2(0)$).

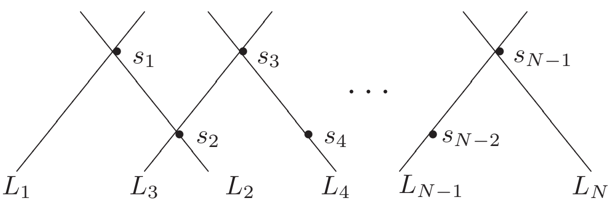
Certainly, there are quite enough transformations to reduce $\mu_2(0)$ to a canonical form consisting of identity and zero blocks only and such that there is exactly one unit in any column of row. Hence, all matrices μ are decomposable, but linear ones. Thus we get the following description of vector bundles on X.

Proposition A.0.1. *Let X be a transversal intersection of two projective lines. Then:*

- *there is an isomorphism of abelian groups* $\mathsf{Pic}_X \xrightarrow{\sim} \mathbb{Z}^2$, $\mathcal{L} \mapsto (d_1, d_2) := \underline{\deg}(\mathcal{L})$;
- *for any* $\mathcal{E} \in \mathsf{VB}_X$ *holds* $\mathcal{E} \cong \bigoplus_{\mathbb{d} \in \mathbb{Z}^2} \mathcal{L}^{r_{\mathbb{d}}}$, *where* $\mathcal{L} \in \mathsf{Pic}_X^{\mathbb{d}}$.

Chains of projective lines (zigzag curves)

Proposition A.0.1 can be easily extended to *chains of projective lines* X :



Here $X = \bigcup_{k=1}^{N} L_k$, the normalization $\widetilde{X} = \bigsqcup_{k=1}^{N} L_k$ and

$$S \cap L_k = \begin{cases} s_k, & \text{if } k = 1; \\ s_{k-1}, s_k, & \text{if } 1 < k < N; \\ s_{k-1}, & \text{if } k = N. \end{cases}$$

Let us choose coordinates on each component L_k such that s_{k-1} corresponds to $(0:1) =: 0$ and s_k corresponds to $(1:0) =: \infty$. Analogously to the case of two crossing lines, it holds $\mathcal{O}_{S,s_k} = \mathbb{k}$, and $\mathcal{M}_{s_k} \cong \mathbb{k}^r$;

$$\mathcal{O}_{\widetilde{S} \cap L_k} = \begin{cases} \mathbb{k}_k(\infty), & \text{if } k = 1; \\ \mathbb{k}_k(0) \oplus \mathbb{k}_k(\infty), & \text{if } 1 < k < N; \\ \mathbb{k}_k(0), & \text{if } k = N. \end{cases}$$

For the first an the last component the trivialization is chosen analogously to the previous example. For each component $L := L_k$, where $1 < k < N$, we use the trivialization:

$$\begin{aligned} \mathcal{O}_L(n) \otimes \mathcal{O}_{L\cap\widetilde{S}} &\xrightarrow{\sim} \mathbb{k}_k(0) \times \mathbb{k}_k(\infty) \\ \zeta \otimes 1 &\longmapsto (\zeta/z_1^n(0), \zeta/z_0^n(\infty)). \end{aligned}$$

Hence, for each component $1 < k < N$ we fix a basis

$$\tilde{\imath}^*(\widetilde{\mathcal{F}}|_{L_k}) \xrightarrow{\cong} \begin{cases} \mathbb{k}_k(\infty)^r, & \text{if } k = 1; \\ \mathbb{k}_k(0)^r \oplus \mathbb{k}_k(\infty)^r, & \text{if } 1 < k < N; \\ \mathbb{k}_k(0), & \text{if } k = N. \end{cases}$$

Matrix problem for a zigzag curve

Hence, the matrix $\tilde{\mu}$ consists of tuple of matrices

$$(\mu_1(\infty), \mu_2(0), \mu_2(\infty), \dots, \mu_N(0)).$$

Matrices $\mu_k(0)$ and $\mu_k(\infty)$ for $1 < k < N$ are simultaneously divided into horizontal blocks according to the structure of $\widetilde{\mathcal{F}}|_{L_k}$. Two horizontal blocks of $\mu_k(0)$ and $\mu_k(\infty)$ with the same weight are called *conjugated*.

The transformation F consists of matrices $\big(F_1 := F_{L_1}, \dots, F_N := F_{L_N}\big)$ in form (2.6), and to obtain $\tilde{\imath}^* F_k$ we should evaluate F_k at preimages of singular points:

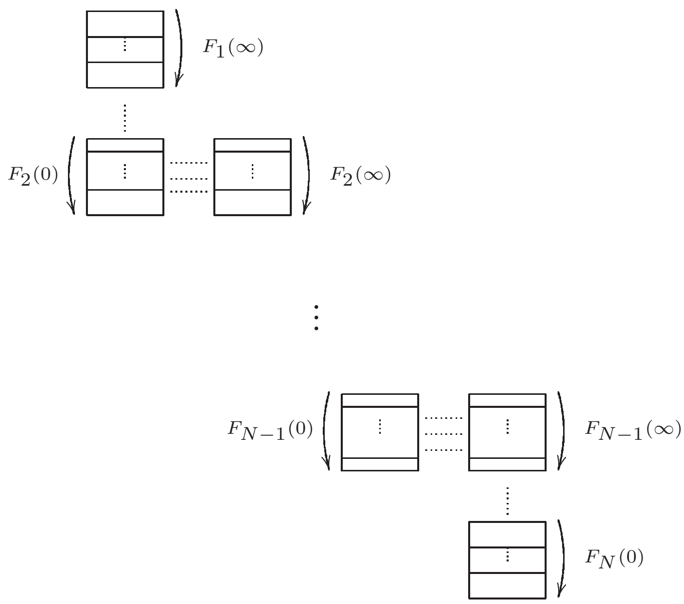
$$\tilde{\imath}^* F_k = \begin{cases} F_k(\infty), & \text{if } k = 1; \\ (F_k(0), F_k(\infty)), & \text{if } 1 < k < N; \\ F_k(0), & \text{if } k = N. \end{cases}$$

where $F_k(0)$ and $F_k(\infty)$ are lower-block-triangular matrices over $\mathbb{k}$, with the same diagonal blocks for $1 < k < N$. The morphism f induces the transformation $\tilde{\pi}^* f = (f_1, \dots, f_{N-1})$ and $f_k \in \mathrm{GL}(\mathbb{k}, r)$.

Hence, we obtaine the matrix problem

$$\begin{cases} \mu_1(\infty) & \mapsto F_1(\infty)\mu_1(\infty) f_1^{-1}, \\ \mu_2(0) & \mapsto F_1(0)\mu_2(0) f_1^{-1}, \\ \mu_2(\infty) & \mapsto F_2(\infty)\mu_2(\infty) f_2^{-1}, \\ \vdots & \\ \mu_{N-1}(0) & \mapsto F_{N-1}(0)\mu_{N-1}(0) f_{N-2}^{-1}, \\ \mu_{N-1}(\infty) & \mapsto F_{N-1}(\infty)\mu_{N-1}(\infty) f_{N-1}^{-1}, \\ \mu_N(0) & \mapsto F_N(0)\mu_N(0) f_{N-1}^{-1}; \end{cases} \tag{A.2}$$

which can be sketched as follows:



We are allowed to perform the following transformations:

1. An arbitrary elementary transformation of columns simultaneously for the matrices $\mu_k(\infty)$ and $\mu_{k+1}(0)$, for $1 \leq k \leq N-1$ Such transformations correspond to f_k.

2. An arbitrary invertible elementary transformation of rows simultaneously for any two conjugated horizontal blocks of the matrices $\mu_k(0)$ and $\mu_k(\infty)$ Such transformations correspond to the diagonal blocks of the matrix F_k.

3. In each of the matrices $\mu_k(0)$ and $\mu_k(\infty)$ we can independently add a scalar multiple of any row with a lower weight to any row with a higher weight. Such transformations correspond to non-diagonal blocks of $F_k(0)$ and $F_k(\infty)$).

Analogously to the previous case the matrix μ can be reduced to the canonical form consisting of zero and identity blocks such that there is a unique unit in each row and each column in any matrix either $\mu_k(0)$ or $\mu_k(\infty)$. Thus all matrices of rank bigger than one are decomposable. Hence, we get the following description of vector bundles, which generalizes the result of Birkoff-Grothendieck on a chain of N projective lines:

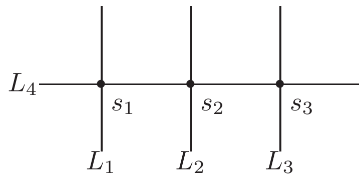
Theorem A.0.2 ([DG01]). *Let X be a chain of N projective lines. Then:*

- *there is an isomorphism of abelian groups $\mathsf{Pic}_X \xrightarrow{\sim} \mathbb{Z}^N$, $\mathcal{L} \mapsto \mathbb{d} := \underline{\deg}(\mathcal{F})$;*

- *for any $\mathcal{E} \in \mathsf{VB}_X$ it holds $\mathcal{E} \cong \bigoplus\limits_{\mathbb{d} \in \mathbb{Z}^N} \mathcal{L}^{r_{\mathbb{d}}}$, where $\mathcal{L} \in \mathsf{Pic}_X^{\mathbb{d}}$.*

Looking at the considered examples one can suggest that a vector bundle on any curve of arithmetic genus zero splits into a direct sum of line bundles. However, this guess is wrong as one can see from the following example. In fact, chains of projective lines are the only curves of arithmetic genus zero, where all indecomposable vector bundles can be classified (see [DG01] Proposition 2.7).

Example A.0.3. Consider the following configuration X:



Here $X = \bigcup\limits_{k=1}^{4} L_k$, the normalization $\widetilde{X} = \bigsqcup\limits_{k=1}^{4} L_k$, $S \cap L_k = s_k$ for $k = 1, 2, 3$ and the component L_4 contains all three singular points $S \cap L_4 = S = \{s_1, s_2, s_3\}$. As in previous examples choose coordinates on each component. Assume that on the component L_4 the preimage of s_1 has coordinates $0 := (0 : 1)$, the point s_1 has coordinates $1 := (1 : 1)$ and s_3 has coordinates $\infty := (1 : 0)$. Choosing trivializations we obtain a matrix problem for

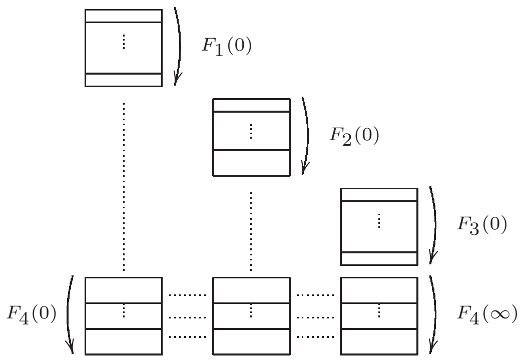
$$\tilde{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4(0), \mu_4(1), \mu_4(\infty)).$$

The transformation F consists of matrices $\big(F_1, F_2, F_3, F_4\big)$ in form (2.6), and to obtain $\tilde{\imath}^* F_k$ we should evaluate F_k at the preimages of singular points, i.e. $\tilde{\imath}^* F_k = (F_k(0))$ for $k = 1, 2, 3$ are lower-block-triangular matrices over $\Bbbk$ and

$$\tilde{\imath}^* F_4 = (F_4(0), F_4(1), F_4(\infty)).$$

All matrices $F_4(0), F_4(1), F_4(\infty)$ have common diagonal blocks, non-diagonal blocks of $F_4(0)$ and $F_4(\infty)$ are independent, and non-diagonal blocks of $F_4(1)$ are some functions of $F_4(0)$ and $F_4(\infty)$. If $\widetilde{\mathcal{F}}$ consists of two blocks of weights c and $c+1$ then the left lower $(F_4(1))_{21}$ is equal to the sum of blocks $(F_4(0))_{21}$ and $(F_4(\infty))_{21}$.

The transformation f induces $\tilde{\pi}^* f = (f_1, f_2, f_3)$ and $f_k \in \mathrm{GL}(\Bbbk, r)$.



We are allowed to perform the following transformations:

1. An arbitrary elementary transformation of columns of matrices simultaneously for matrices μ_1 and $\mu_4(0)$; μ_2 and $\mu_4(1)$; μ_3 and $\mu_4(\infty)$. Such transformations correspond to the matrices f_k.

2. An arbitrary invertible elementary transformation of rows inside of any horizontal blocks of μ_k for $k = 1, 2, 3$. Such transformations correspond to diagonal blocks of F_k.

3. An arbitrary invertible elementary transformation of rows inside horizontal blocks simultaneously of any triple of conjugated horizontal blocks of $\mu_4(0)$, $\mu_4(\infty)$ and $\mu_4(1)$. Such transformations correspond to diagonal blocks of F_4.

4. For each of the matrices μ_k ($k = 1, 2, 3$) we can add a scalar multiple of any row with a lower weight to any row with a higher weight Such transformations correspond to the non-diagonal blocks of F_k.

5. We can simultaneously in $\mu_4(0)$ and $\mu_4(1)$ or $\mu_4(\infty)$ and $\mu_4(1)$ add a scalar multiple of any row with a lower weight to any row with a higher weight. Such transformations correspond to non-diagonal blocks of $F_4(0)$ and $F_4(\infty)$.

For such a problem we can easily describe line bundles, but in generic case this problem is wild (see [DG01] for a proof).

The situation changes if we consider simple vector bundles. Namely, as a consequence of Lemma 2.6.4 for curves of arithmetic genus zero the following statement holds true:

Theorem A.0.4. *Let X be a curve of arithmetic genus zero with N components. Then*

$$\mathsf{VB}^s_X = \mathsf{Pic}(X) \cong \mathbb{Z}^N.$$

Proof. Decompose X onto components and apply Lemma 2.6.4 inductively. $\square$

Appendix B

Representation types of bocses

Here we recall the definitions of representation types and Drozd's Tame and Wild Theorem.

Finite and discrete types

Let $\mathscr{A}$ be a bocs. We say that the representation type of $\mathscr{A}$ is *finite* if $\mathsf{Rep}(\mathscr{A})$ contains finitely many isomorphism classes of indecomposable objects.

The representation type of $\mathscr{A}$ is said to be *discrete* if for each vector dimension $\mathbb{d}$ the set of isomorphism classes of indecomposable objects of dimension $\mathbb{d}$ is finite.

Tame type

Following [Dro01] we extend the notion of a representation a bocs. Let $\mathscr{A}$ be a bocs and C be a linear category.

Definition B.0.5. A *representation of $\mathscr{A}$ over C* is $\mathbb{k}$-linear functor

$$M : \mathscr{A} \to \mathsf{Pr}(C),$$

where $\mathsf{Pr}(C)$ is a category of finitely dimensional right projective C-modules.

The category of representations over C is denoted by $\mathsf{Rep}(\mathscr{A}, C)$.

Definition B.0.6. A representation $M \in \mathsf{Rep}(\mathscr{A}, C)$ is called *strict* (or equivalently they say the functor $-\otimes_C M$ is a *representation embedding*) if for arbitrary representations N and $N' \in \mathsf{Rep}(C)$ we have

- $N \otimes_C M \cong N' \otimes_C M$ implies $N \cong N'$,
- if N is indecomposable, then so is $N \otimes_C M$.

If moreover the functor $- \otimes_C M$ is dense then it is called a *representation equivalence.*

Let $R_f := \Bbbk[t, f(t)^{-1}]$ be a localization of the rational algebra $\Bbbk[t]$ by a polynomial f. The category corresponding to this algebra we call a *rational category.* Note that $ind(\mathsf{Rep}(R_f))$ consists of Jordan blocks $J_n(\lambda)$ with eigenvalues λ not a root of f.

If $M \in \mathsf{Rep}(\mathscr{A}, R_f)$ is a strict representation, then the set

$$\mathbb{F}(M) = \{N \otimes_{R_f} M | N \in \mathsf{Rep}(R_f)\}$$

is called a *one-parameter family of representations* or a *rational family.*

Definition B.0.7. A bocs $\mathscr{A}$ is said to be *(representation) tame* if there exists a set of representations $\mathfrak{M}$ such that:

- each indecomposable representation in $M \in \mathfrak{M}$ is strict of finite rank over some rational algebra $R_f(M)$ (which probably depends on M);
- the set $\mathfrak{M}$ splits into strata :

  $$\mathfrak{M} = \bigcup_{\mathbb{d} \in \mathbb{Z}^n_{\geq 0}} \mathfrak{M}_{\mathbb{d}},$$

  where each stratum $\mathfrak{M}_{\mathbb{d}}$ is a finite set of representations of finite vector dimension $\mathbb{d}$;
- for each vector dimension $\mathbb{d}$

  $$ind(\mathsf{Rep}(\mathscr{A}))(\mathbb{d}) \setminus S_{\mathbb{d}} = \bigcup_{M \in \mathfrak{M}_{\mathbb{d}}} \mathbb{F}(M),$$

  where $S_{\mathbb{d}}$ is a finite set of indecomposable representations of vector dimension $\mathbb{d}$. It means that almost all representations belong to some rational family.

The set $\mathfrak{M}$ is called a *parameterizing set of representations* of the bocs $\mathscr{A}$.

Brick-tame type

Absolutely analogously we can define a brick-tame bocs.

Definition B.0.8. A representation $M \in \mathsf{Rep}(\mathscr{A}, C)$ is called *brick-strict* (or equivalently the functor $- \otimes_C M$ is called a *brick-embedding*) if for arbitrary representations N and $N' \in \mathsf{Rep}(C)$ we have

- $N \otimes_C M \cong N' \otimes_C M$ implies $N \cong N'$,

- if N is a brick, then so is $N \otimes_C M$.

If moreover the functor $- \otimes_C M$ is dense on bricks then it is called a *brick-equivalence.*
If $M$is a brick-strict representation of $\mathscr{A}$ over a rational algebra R_f, then the set

$$\mathbb{F}^s(M) = \{N \otimes_{R_f} M | N \in \mathsf{Br}(R_f)\}$$

is called a *one-parameter family of bricks* or *brick-rational family.*

Definition B.0.9. A Roiter bocs $\mathscr{A}$ is said to be *(representation) brick-tame* if there is a set of representations $\mathfrak{M}$ such that:

- each representation in $M \in \mathfrak{M}$ is brick-strict of finite rank over some rational algebra $R_f(M)$;
- the set $\mathfrak{M}$ splits into strata :

$$\mathfrak{M} = \bigcup_{\mathbb{d} \in \mathbb{Z}^n_{\geq 0}} \mathfrak{M}_{\mathbb{d}},$$

  where each stratum $\mathfrak{M}_{\mathbb{d}}$ is a finite set;
- for each vector dimension $\mathbb{d}$ we have

$$\mathsf{Br}(\mathbb{d}) \setminus S_{\mathbb{d}} = \bigcup_{M \in \mathfrak{M}_{\mathbb{d}}} \mathbb{F}^s(M),$$

  where $S_{\mathbb{d}}$ is a finite set of bricks of indecomposable representations of vector dimension $\mathbb{d}$. (i.e. it means that almost all bricks belong to some rational family.)

The set $\mathfrak{M}$ is called a *parameterizing set of bricks* of the bocs $\mathscr{A}$.

Wild type

Definition B.0.10. A Roiter bocs $\mathscr{A}$ is called *representation wild* if for any finitely generated algebra C there is a strict representation $M \in \mathsf{Rep}(\mathscr{A}, C)$.

Theorem B.0.11 (Drozd's Tame and Wild Theorem [Dro79]). *A representation infinite Roiter bocs is either tame or wild.*

Brick-wild type

There is a conjecture saying that a statement analogous to Theorem B.0.11 should hold for bricks. However, we need a proper definition of brick-wildness. We hope the following one fits well, but at the moment we can not claim it rigorously.

Definition B.0.12. A Roiter bocs $\mathscr{A}$ is called *brick-wild* if for any finitely generated algebra C there is a representation $M \in \mathsf{Rep}(\mathscr{A}, C)$, such that $-\otimes_C M$ is a brick-embedding.

Appendix C

Bimodule problems

Let A be a category with a finite set of indecomposable objects I, and U be an A-bimodule. The definitions and the theorem below hold even in a general set up, but for the sake of clarity we assume A and U finite dimensional as in Chapter 6.

Define the new category $\mathsf{El}(U)$ of *elements of the bimodule* U or matrices over U, as follows:

- Objects are pairs $(\bar{\imath}, u)$, where $\bar{\imath} = \bigoplus_{i \in I} i^{\alpha_i}$, and $u \in U(\bar{\imath}, \bar{\imath})$,
- Morphisms from $u \in U(\bar{\imath}, \bar{\imath})$ to $u' \in U(\bar{\imath}', \bar{\imath}')$ are morphisms $a \in A(\bar{\imath}, \bar{\imath}')$ such that
$$au = u'a.$$

Remark C.0.13. Some authors call the category $\mathsf{El}(U)$ the category of *representations of the bimodule*, but we do not use this notation in order to avoid the confusion with representations of bocses.

The problem of describing isomorphism classes of indecomposable objects of $\mathsf{El}(U)$ is called a *bimodule problem*. Bimodule problems form a big class of problems in which we are interested.

To an A-bimodule U we can assign a biquiver $Q = (Q_0, Q_1)$ on the set of objects I, where

- Q_0 is the set of solid arrows,
- Q_1 is the set of dotted arrows,
- the set of arrows $Q_0(i,j)$ is a basis of $U(i,j)$ and any path with exactly one solid arrow from i to j is a linear combination of arrows from $Q_0(i,j)$,
- the set of arrows $Q_1(i,j)$ is a basis of the *radical* [1] $\mathsf{rad}(A)(i,j)$ and any dotted path from i to j is a linear combination of arrows from $Q_1(i,j)$.

[1]recall that the radical of A is defined to be the ideal of all noninvertible morphisms of A.

Such a biquiver is called a *base* of the A-bimodule U. The base is called *multiplicative* if for any path xy of length 2 with at most one solid arrow such that $xy \neq 0$, there is a unique arrow of $z \in Q$, which is dotted if both x and y are dotted and solid otherwise, such that $z = x \cdot y$. Not all bimodules have a multiplicative base.

Let z be an arrow of a multiplicative base Q of an A-bimodule U. The *level* of an arrow z is the number of arrows in the longest path $x_n \dots x_2 x_1$ such that $z = x_n \dots x_2 x_1$. (Note that such a path contains at most one solid arrow.) In terms of bimodules the level of a generator z means the maximal power of the radical which contains z, (the radical of a bimodule U if the path contains a solid arrow, and the radical of A otherwise.) Note that if $z = xy$ then the levels of x and y are strictly smaller than the level of z.

Bimodule problems in terms of bocses

Theorem C.0.14 ([Dro79],[Dro01]). *Let A be a category, then for any A-bimodule U there exists a normal free triangular and linear bocs $\mathscr{B} = (B, W)$ such that $\mathsf{Rep}(\mathscr{B})$ is equivalent to $\mathsf{El}(U)$.*

For the sake of clarity we give the proof only for bimodules with multiplicative base. The proof in the general case can be found in [Dro01] or [Dro79]. It is almost the same with the difference that instead of the multiplicative base one should take bases of $\mathsf{rad}(A)^n(i,j) \bmod \mathsf{rad}(A)^{n+1}(i,j)$ and $\mathsf{rad}(U)^n(i,j) \bmod \mathsf{rad}(U)^{n+1}(i,j)$, and bases of the dual spaces $D\,\mathsf{rad}(A)(i,j)$, $D\,\mathsf{rad}(U)(i,j)$ for $i, j \in I$, where $\mathsf{rad}(U)^1 = \mathsf{rad}(A)U + U\,\mathsf{rad}(A)$ and $\mathsf{rad}(U)^{n+1} = \mathsf{rad}(A)U^n + U^n\,\mathsf{rad}(A)$. The key point is that there exists N such that for all $n > N$ $\mathsf{rad}(A)^n = 0$.

Proof. If the A-bimodule U has a multiplicative base $Q = (Q_0, Q_1)$ then the proof can be rewritten as follows.

An A-bimodule is completely defined by its basis biquiver Q and multiplication lows:

$$\begin{aligned} A(k,j) \times A(i,k) &\longrightarrow A(i,j) \\ A(k,j) \times U(i,k) &\longrightarrow U(i,j) \\ U(k,j) \times A(i,k) &\longrightarrow U(i,j) \end{aligned}$$

for all objects $i \in I$ and vice versa. The multiplication laws can be given as a system of relations for corresponding paths of length 2. For a pair of arrows (x, y) from either $Q_1(k,j) \times Q_1(i,k)$ or $Q_1(k,j) \times Q_0(i,k)$ or $Q_0(k,j) \times Q_1(i,k)$

define the set of relations:

$$x \cdot y = \sum_{z \in Q_t(i,j)} \lambda(x,y,z)z,$$

where $t = 1$ in case $x, y \in Q_1$ and $t = 0$ otherwise. Thus an A-bimodule U is a biquiver Q together with the set of $\lambda(x,y,z) \in \mathbb{Z}_2$.

To a biquiver Q we can assign a differential ∂ by the following rule: for a solid arrow $b \in Q_0(i,j)$ define:

$$\partial b = \sum_{k \in I} \Big(\sum_{\substack{c \in Q_0(k,i) \\ v \in Q_1(j,k)}} \lambda(c,v,b)cv - \sum_{\substack{c \in Q_0(j,k) \\ v \in Q_1(k,i)}} \lambda(v,c,b)vc \Big)$$

and for a dotted arrow $v \in Q_1(i,j)$ define

$$\partial v = \sum_{k \in I} \Big(\sum_{\substack{w \in Q_1(k,i) \\ u \in Q_1(j,k)}} \lambda(u,w,v)uw \Big).$$

By the Leibniz rule (D3) prolong ∂ to the whole $\Bbbk Q$. The constructed ∂ is indeed a differential: the conditions (D1)-(D3) and (D5*) are satisfied automatically . Define the level function $h : Q \to \mathbb{N}$. For $z \in Q$ let $h(z)$ be the level of z i.e. the number of arrows in the longest path. Then we get the triangularity condition (D4), since from the equality $z = xy$ follows $h(z) > h(x)$ and $h(z) > h(y)$.

The construction of the bocs $\mathscr{B} = (B, V)$ implies the equivalence of the categories $\mathsf{Rep}(\mathscr{B})$ and $\mathsf{El}(U)$, which completes the proof. ☐

Remark C.0.15. Let us stress that contrary to the definition of a biquiver associated to a bimodule: for a free bocs the basis of $A(i,j) \cong \Bbbk Q_0(i,j)$ is formed not only by arrows but by all *paths* of degree 0 from i to j. Analogously, the basis of $\overline{V}(i,j)$ consists not only of dotted arrows but of all paths of degree one form i to j.

Appendix D

Matrix reduction and Fourier-Mukai transforms

Let E be a projective Gorenstein variety over an algebraically closed field $\Bbbk$, $\mathcal{D}^b(\mathsf{Coh}_E)$ the bounded derived category of coherent sheaves on E and Perf_E its full subcategory of perfect complexes. Recall that a complex $\mathcal{E}$ is perfect if it is isomorphic in $\mathcal{D}^b(\mathsf{Coh}_E)$ to a bounded complex of locally free sheaves on E.

Definition D.0.16 (Definition 2.5 in [ST01]). For a perfect complex $\mathcal{E}$ the twist functor

$$\mathsf{T}_{\mathcal{E}} : \mathcal{D}^b(\mathsf{Coh}_E) \longrightarrow \mathcal{D}^b(\mathsf{Coh}_E)$$

is defined by the rule

$$\mathsf{T}_{\mathcal{E}}(\mathcal{F}) = \mathsf{Cone}\big(\mathrm{RHom}(\mathcal{E}, \mathcal{F}) \overset{\Bbbk}{\otimes} \mathcal{E} \xrightarrow{ev} \mathcal{F}\big),$$

where $\mathrm{RHom}(\mathcal{E}, \mathcal{F}) \overset{\Bbbk}{\otimes} \mathcal{E} := \bigoplus\limits_{i \in \mathbb{Z}} \mathcal{E}[-i]^{n_i(\mathcal{E},\mathcal{F})}$ and $n_i(\mathcal{E}, \mathcal{F}) = \dim_{\Bbbk}\big(\mathrm{Hom}(\mathcal{E}, \mathcal{F}[i])\big)$.

Remark D.0.17. Since taking cones in a triangulated category is not a functorial operation, the action of $\mathsf{T}_{\mathcal{E}}$ on morphisms is defined using the language of *enhanced* triangulated categories, see [ST01, Chapter 2] for more details.

Remark D.0.18. Note that for a perfect complex $\mathcal{E}$ we also obtain a functor

$$\mathsf{T}_{\mathcal{E}} : \mathsf{Perf}_E \longrightarrow \mathsf{Perf}_E,$$

i.e. the twist functor restricts to the full subcategory of perfect complexes.

By the definition of $\mathsf{T}_{\mathcal{E}}$, for any object $\mathcal{F} \in \mathcal{D}^b(\mathsf{Coh}_E)$ we have an exact triangle

$$\mathrm{RHom}(\mathcal{E}, \mathcal{F}) \overset{\Bbbk}{\otimes} \mathcal{E} \xrightarrow{ev} \mathcal{F} \longrightarrow \mathsf{T}_{\mathcal{E}}(\mathcal{F}) \xrightarrow{+} .$$

For a triangulated category D let $K(\mathsf{D})$ be the K-group of D. In particular, write

$$K_0(E) := K\big(\mathcal{D}^b(\mathsf{Coh}_E)\big) \cong K(\mathsf{Coh}_E)$$

and

$$K^0(E) := K\big(\mathsf{Perf}_E\big) \cong K(\mathsf{VB}(E)) \cong \mathbb{Z} \oplus \mathsf{Pic}(E),$$

see [Gro77]. Since the twist functor $\mathsf{T}_{\mathcal{E}}$ is exact, it induces group homomorphisms

$$[\mathsf{T}_{\mathcal{E}}] : K_0(E) \longrightarrow K_0(E) \quad \text{and} \quad [\mathsf{T}_{\mathcal{E}}] : K^0(E) \longrightarrow K^0(E),$$

given by the rule $[\mathsf{T}_{\mathcal{E}}]([\mathcal{F}]) = [\mathcal{F}] - \langle [\mathcal{E}], [\mathcal{F}] \rangle [\mathcal{E}]$, where

$$\langle [\mathcal{E}], [\mathcal{F}] \rangle = \sum_{i \in \mathbb{Z}} \dim_{\mathbb{k}} \operatorname{Hom}(\mathcal{E}, \mathcal{F}[i])$$

is the Euler form.

Definition D.0.19 (Definition 2.9 in [ST01]). Let E be a projective Calabi-Yau variety of dimension n, i.e. E is Gorenstein and $\omega_E \cong \mathcal{O}_E$. An object $\mathcal{E} \in \mathsf{Perf}_E$ is called spherical if

$$\operatorname{Hom}(\mathcal{E}, \mathcal{E}[i]) = \begin{cases} \mathbb{k}, & \text{if } i = 0, n \\ 0, & \text{if } i \neq 0, n. \end{cases}$$

Theorem D.0.20 (Proposition 2.10 in [ST01]). *Let E be a projective Calabi-Yau variety of dimension n, $\mathcal{E} \in \mathsf{Perf}_E$ a spherical object. Then the twist functors*

$$\mathsf{T}_{\mathcal{E}} : \mathcal{D}^b(\mathsf{Coh}_E) \longrightarrow \mathcal{D}^b(\mathsf{Coh}_E) \quad \textit{and} \quad \mathsf{T}_{\mathcal{E}} : \mathsf{Perf}_E \longrightarrow \mathsf{Perf}_E$$

are equivalences of triangulated categories.

Remark D.0.21. In the case of elliptic curves and weighted projective lines of tubular type twist functors were independently discovered by Lenzing and Meltzer under the name *tubular mutations*, see ref [24] and [25] in [BK3].

From now on let E be a *reduced Gorenstein projective curve of arithmetic genus one*. Note that we automatically obtain: $\omega_E \cong \mathcal{O}_E$.

Proposition D.0.22 (Formula 3.11 in [ST01]). *Let $p \in E$ be a smooth point, then the skyscraper sheaf $\mathbb{k}(p)$ is spherical. Moreover, there is an isomorphism of functors: $T_{\mathbb{k}(p)} \cong \mathcal{O}(p) \otimes -$.*

Another natural example of a spherical object in $\mathcal{D}^b(\mathsf{Coh}_E)$ is the structure sheaf $\mathcal{O}$. The following lemma follows from the definition of twist functors.

Lemma D.0.23. *Let $\mathcal{F} \in \mathsf{Coh}_E$ be a coherent sheaf such that $H^1(\mathcal{F}) = 0$. Then*

$$\mathsf{T}_{\mathcal{O}}(\mathcal{F}) = \big(\ldots \longrightarrow 0 \longrightarrow H^0(\mathcal{F}) \otimes \mathcal{O} \xrightarrow{ev} \mathcal{F} \longrightarrow 0 \longrightarrow \ldots\big).$$

In particular, let p be a point of E (either smooth or singular) and $\mathcal{I}_p$ its ideal sheaf, then

$$\mathsf{T}_{\mathcal{O}}(\Bbbk(p)) \cong \mathcal{I}_p[1].$$

Seidel and Thomas also observed that under certain conditions twist functors satisfy braid group relations.

Proposition D.0.24 (Proposition 2.13 in [ST01]). *Let $p_0 \in E$ be an arbitrary smooth point, then there is the following isomorphism of functors:*

$$\mathsf{T}_{\mathcal{O}}\, \mathsf{T}_{\Bbbk(p_0)}\, \mathsf{T}_{\mathcal{O}} \cong \mathsf{T}_{\Bbbk(p_0)}\, \mathsf{T}_{\mathcal{O}}\, \mathsf{T}_{\Bbbk(p_0)}\,.$$

Theorem D.0.25 (Theorem 3.13 in [Mu81]; Lemma 3.2 and Section 3.4 in [ST01]; Remark 5.13 in [BK3]). *Let E be an irreducible plane cubic curve, $p_0 \in E$ a smooth point, then there is an isomorphism of functors:*

$$\mathsf{T}_{\Bbbk(p_0)}\, \mathsf{T}_{\mathcal{O}}\, \mathsf{T}_{\Bbbk(p_0)} \cong \mathbb{F}[-1],$$

where $\mathbb{F} = \mathbb{F}^{\mathcal{P}}$ is the Fourier-Mukai transform *with the Fourier kernel*

$$\mathcal{P} = \mathcal{I}_\Delta \otimes \pi_1^*\mathcal{O}(p_0) \otimes \pi_2^*\mathcal{O}(p_0) \in \mathsf{Coh}(E \times E).$$

Moreover, $\mathbb{F}^2 \cong i^*[1]$, where $i : E \longrightarrow E$ is the involution of E such that $i(p_0) = p_0$ and

$$\mathrm{Aut}\big(\mathcal{D}^b(\mathsf{Coh}_E)\big) = \langle \mathsf{T}_{\mathcal{O}}, \mathsf{T}_{\Bbbk(p_0)}, [1], \mathrm{Aut}(E)\rangle.$$

generated as a group. Finally, if $\mathcal{E} \in \mathcal{D}^b(\mathsf{Coh}_E)$ is a spherical object, then $\mathcal{E}$ is isomorphic to a shift either of a simple vector bundle or of the structure sheaf of a smooth point of E. In particular, if $\mathcal{E}$ is a simple vector bundle and $\mathbb{G} \in \mathrm{Aut}\big(\mathcal{D}^b(\mathsf{Coh}_E)\big)$ then $\mathbb{G}(\mathcal{E})$ is isomorphic to a shift of a simple coherent sheaf.

Remark D.0.26. Let E be a *singular* irreducible cubic curve, i.e. E is either nodal or cuspidal, then

$$K_0(E) = \langle \mathcal{O}, \Bbbk(p_0)\rangle \cong \mathbb{Z}^2.$$

Moreover, for any coherent sheaf $\mathcal{F}$: $[\mathcal{F}] = \mathrm{rk}(\mathcal{F})[\mathcal{O}] + \chi(\mathcal{F})[\Bbbk(p_0)]$. Following the terminology of the homological mirror symmetry, let

$$\tau(\mathcal{F}) = \big(\mathrm{rk}(\mathcal{F}), \deg(\mathcal{F})\big) \in \mathbb{Z}^2$$

be the *topological charge* of $\mathcal{F}$. Recall that in the basis $\{[\mathcal{O}], [\Bbbk(p_0)]\}$ the action of the twist functors $\mathsf{T}_{\mathcal{O}}$ and $\mathsf{T}_{\Bbbk(p_0)}$ is given by the matrices

$$[\mathsf{T}_{\mathcal{O}}] = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad [\mathsf{T}_{\Bbbk(p_0)}] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

In particular, let $\mathcal{E}$ be a simple vector bundle and $\tau([\mathcal{E}]) = (r, d)$ then

$$\tau([\mathsf{T}_{\mathcal{O}}(\mathcal{E})]) = (r - d, d) \quad \text{and} \quad \tau([\mathsf{T}_{\Bbbk(p_0)}(\mathcal{E})]) = (r, d + r).$$

A similar result holds for a smooth elliptic curve, when we replace $K_0(E)$ by the so-called *numerical* K–group $K_0(E)/\mathrm{rad}\langle\ ,\ \rangle$, where $\langle\ ,\ \rangle$ is the Euler form on $K_0(E)$.

Let $E = \cup_{k=1}^{N} L_k$ be a reduced rational projective curve of arithmetic genus one with N irreducible components (so $L_k \cong \mathbb{P}^1$ for $k = 1, \dots, N$) and $\pi : \widetilde{E} \longrightarrow E$ its normalization. Then we have N different finite maps

$$\pi_k : \mathbb{P}^1 = L_k \longrightarrow \widetilde{E} \xrightarrow{\pi} E$$

and one can define N additive homomorphisms $\deg_k : K^0(E) \longrightarrow \mathbb{Z}$ given by the rule

$$\deg_k([\mathcal{E}]) := \deg(\pi_k^*[\mathcal{E}]).$$

Similarly to the case of irreducible curves we define the map

$$\tau = (\mathrm{rk}, \deg_1, \dots, \deg_N) : \quad K^0(E) \longrightarrow \mathbb{Z}^{N+1}.$$

For each irreducible component L_k choose a smooth point $p_k \in L_k$ and consider the group $G := \langle \mathsf{T}_{\mathcal{O}}, \mathsf{T}_{\Bbbk(p_1)}, \dots, \mathsf{T}_{\Bbbk(p_N)} \rangle \subseteq \mathrm{Aut}\big(\mathcal{D}^b(\mathsf{Coh}_E)\big)$. By the definition, G acts on the abelian group $K^0(E)$. However, our aim is to construct a representation of G on $\mathrm{im}(\tau) \cong \mathbb{Z}^{N+1}$.

Lemma D.0.27. *In the notation above we have: $G\big(\ker(\tau)\big) \subseteq \ker(\tau)$. In particular, there exists a natural group homomorphism $G \longrightarrow \mathrm{GL}\big(\mathrm{im}(\tau)\big) = \mathrm{GL}_{N+1}(\mathbb{Z})$.*

Proof. Since we have the isomorphism $(\mathrm{rk}, \det) : K^0(E) \longrightarrow \mathbb{Z} \oplus \mathsf{Pic}(E)$, one can show that an arbitrary element $x \in K^0(E)$ can be written in the form

$$x = r[\mathcal{O}] + \sum_{p \in E_{reg}} n_p[\Bbbk(p)],$$

where all coefficients n_p but finitely many are zero. If $\tau(x) = 0$ then $r = 0$ and moreover

$$\sum_{p \in L_k} n_p = 0, \quad \text{for any} \quad k = 1, \dots, N.$$

The statement follows from the isomorphisms $\mathsf{T}_{\Bbbk(p)}(\Bbbk(q)) \cong \Bbbk(q)$ and $\mathsf{T}_{\mathcal{O}}(\Bbbk(q)) \cong \mathcal{I}_q[1]$. □

Corollary D.0.28. *Fix the basis* $\{[\mathcal{O}], [\Bbbk(p_1)], \dots, [\Bbbk(p_N)]\}$ *of the group* $K^0(E)/\ker(\tau) \cong \mathbb{Z}^{N+1}$. *Then for any vector bundle* $\mathcal{E} \in \mathsf{VB}(E)$

$$[\mathcal{E}] = \mathrm{rk}(\mathcal{E})[\mathcal{O}] + \sum_{k=1}^{N} \deg(\mathcal{E}|_{L_k})[\Bbbk(p_k)]$$

and the action of G *on* $K^0(E)/\ker(\tau)$ *is given by the matrices:*

$$[\mathsf{T}_{\mathcal{O}}] = \begin{pmatrix} 1 & -1 & -1 & \dots & -1 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad \textit{and} \quad [\mathsf{T}_{\Bbbk(p_k)}] = \begin{array}{c} \\ \\ \\ k \\ \\ \\ \end{array} \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & \dots & 0 \\ 1 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{pmatrix}.$$

This means that if $\tau(\mathcal{E}) = (r, \mathbb{d}) = (r, d_1, \dots, d_N)$ then

$$\tau\big([\mathsf{T}_{\mathcal{O}}(\mathcal{E})]\big) = (r - d, d_1, \dots, d_N),$$

where $d = d_1 + \dots + d_N = \deg(\mathcal{E}) = \chi(\mathcal{E})$, and

$$\tau\big([\mathsf{T}_{\Bbbk(p_k)}(\mathcal{E})]\big) = (r, d_1, \dots, d_{k-1}, d_k + r, d_{k+1}, \dots, d_N).$$

Example D.0.29. Let E be a reduced rational curve of arithmetic genus one with two irreducible components, i.e a Kodaira fiber of type I_2 or III then the group $G = \langle \mathsf{T}_{\mathcal{O}}, \mathsf{T}_{\Bbbk(p_1)}, \mathsf{T}_{\Bbbk(p_2)} \rangle$ acts on $K^0(E)/\ker(\tau) \cong \mathbb{Z}^3$ as

$$[\mathsf{T}_{\mathcal{O}}] = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [\mathsf{T}_{\Bbbk(p_1)}] = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [\mathsf{T}_{\Bbbk(p_2)}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Observe that the group G is generated by the semigroup of the *brick-reduction automaton* 7.5.2 of the corresponding matrix problem.

This observation leads to the following natural

Question: Let E be a Kodaira fiber either of type I_N, $N \geq 1$ or of type II, III or IV. Can the matrix reduction functor

$$\mathsf{VB}^s_E(r, \mathbb{d}) \longrightarrow \mathsf{VB}^s_E(r', \mathbb{d}')$$

be lifted to a functor

$$\mathbb{G} : \mathcal{D}^b(\mathsf{Coh}_E) \longrightarrow \mathcal{D}^b(\mathsf{Coh}_E)$$

generated by the spherical twists $\mathsf{T}_{\mathcal{O}}, \mathsf{T}_{\Bbbk(p_1)}, \dots, \mathsf{T}_{\Bbbk(p_N)}$?

This question is closely related with other open problems about $\mathcal{D}^b(\mathsf{Coh}_E)$.

Q1: What is the classification of the spherical objects in $\mathcal{D}^b(\mathsf{Coh}_E)$?

Q2: Does the group G acts transitively on the set of spherical objects?

Q3: What is the kernel of the group homomorphism $\mathrm{Aut}\big(\mathcal{D}^b(\mathsf{Coh}_E)\big) \longrightarrow \mathrm{GL}_{N+1}(\mathbb{Z})$?

Q4: Is it true that $\mathrm{Aut}\big(\mathcal{D}^b(\mathsf{Coh}_E)\big) = \langle \mathrm{Aut}(E), [1], \mathsf{T}_{\mathcal{O}}, \mathsf{T}_{\Bbbk(p_1)}, \dots, \mathsf{T}_{\Bbbk(p_N)} \rangle$?

We hope to come back to these questions in a future work.

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